Stress Concentration in a Nonlinear – Elastic Plane with a Circular Hole

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Резюме: *Класът от материали изследван в представената статия е известен с наименованието хармоничен. В работата е конструиран дискратен алгоритъм за определяне на коефициента на концентрация на напреженията в еластична равнина с кръгов отвор. Задчате е решена с аналитични методи.*

Ключови думи: *Нелинейна теория на еластичността, метод на аналитичното продължение, концентрация на напреженията*

INTRODUCTION AND STATEMENT OF THE PROBLEM

In most of the elastic-static problems, the main subject is that of determining the stress concentrations in the critical points of the studied solid. It is known that if there's a hole in the solid, its critical points lie on the hole's contour; inside the elastic medium the stress concentrations decreases. This problem has been solved long ago in the problems of linear theory plane problems [5, 6]. Although the solution has a substantial disadvantage. The solution does net depend (for certain reasons) on the elastic properties of the material, which on its own arouse suspicion about the correspondence between the obtained theoretical results and reality.

The present article suggests a method for solving the problem for nonlinear elastic material of harmonic type [3], as the problem of determining the coefficient of stress concentration in the contour points of an infinite elastic plain containing circular hole when there's perfectly solid driving wheel soldered in the hole is examined particularly.

The material named by John [3] harmonic and named by Lure [4],is characterized by the setting of the function of the plane – strain elastic potential, which is represented in the following way in the two-dimensional case.

$$w=\frac{1}{2}\left[\lambda(\delta_1+\delta_2)^2+2\mu\left(\delta_1^2+\delta_2^2\right)\right], \tag{1}$$

where λ and μ are Lame's elastics constants and δ_1 and δ_2 are the principle stretch ration. Formally the function W Is described in a way, analogous to the respective function for Hock's classical material. But there's a substantial difference lying in the fact that in the represented case it appears to nonlinear function of the elastic removals gradients.

$$\delta_1=\sqrt{1+2\frac{\partial v}{\partial x}+\left[\frac{\partial u}{\partial x}\right]^2+\left[\frac{\partial v}{\partial x}\right]^2}-1,\quad \delta_2=\sqrt{1+2\frac{\partial v}{\partial y}+\left[\frac{\partial u}{\partial y}\right]^2+\left[\frac{\partial v}{\partial y}\right]^2}-1 \tag{2}$$

Here *u* and *v* are components of the displacement vector. Nonlinear representation of the field of elastic elements by two analytical functions $\varphi(z)$ and $\psi(z)$in studied area in a result of (1) and (2).

$$c(\sigma_1+\sigma_2)+4\mu=\frac{\lambda+2\mu}{\sqrt{J}}q\Omega(q),\qquad c(\sigma_2-\sigma_1-2ir_{12})=-\frac{4(\lambda+2\mu)}{\sqrt{J}}\frac{\Omega(q)}{q}\frac{\partial z^*}{\partial z}\frac{\partial z^*}{\partial \bar{z}}, \tag{3}$$

$$\frac{\partial z^*}{\partial z}=\frac{\mu}{\lambda+2\mu}\varphi'^2(z)+\frac{\lambda+\mu}{\lambda+2\mu}\frac{\varphi'(z)}{\overline{\varphi'(z)}},\qquad \frac{\partial z^*}{\partial \bar{z}}=-\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)\overline{\varphi'(z)}}{\varphi'^2(z)}-\overline{\psi'(z)}\right] \tag{4}$$

$$u+i\upsilon=\frac{\mu}{\lambda+2\mu}\int\varphi'^2(z)dz+\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)}{\overline{\varphi'(z)}}+\overline{\psi(z)}\right]-z,\quad z^*=z+u+v \tag{5}$$

$$\sqrt{J}=\frac{\partial z^*}{\partial z}\frac{\partial \bar{z}^*}{\partial \bar{z}}-\frac{\partial z^*}{\partial \bar{z}}\frac{\partial \bar{z}^*}{\partial z},\quad q=2\left|\frac{\partial z^*}{\partial z}\right|,\quad \Omega(q)=q-2+\frac{2\mu-\lambda(c-1)}{\lambda+2\mu}, \tag{6}$$

$$C=\begin{cases}1 & \text{In the case of a plane deformation}\\ \dfrac{3\lambda+2\mu}{\lambda+2\mu}-\dfrac{\lambda q}{\lambda+2\mu} & \text{In the case of a general plane strained state}\end{cases} \tag{7}$$

In these equations σ_1, σ_1 τ_{12} are components of the Cauchy stress tensor. To (3) – (7) we will add the results got in work [2] where the using of the method of analytical prolongation enables the elastic elements to be represented just by only one piece –wise analytical function, which turns out to be suitable in solving problems and in doing certain calculations. We will write down the functional equation and the nonlinear boundary problem for the case in which on the boundary are known elastic removals only.

$$\frac{\mu}{\lambda+\mu}\varphi'^2(z)-\frac{1}{2\pi i}\int_L\frac{\varphi^-(t)dt}{\overline{\varphi^-(t)}(t-z)^2}=\frac{\mu a_0^2}{\lambda+\mu}+\frac{\overline{b_0}}{z^2}-\frac{1}{2\pi i}\int_L\frac{g(t)dt}{t-z} \tag{8}$$

$$\frac{\mu}{\lambda+\mu}\varphi'^{\pm2}(t)+\varphi'^{\mp2}(t)+\frac{d}{dt}\left[\frac{\varphi^{\pm}(t)-\varphi^{\mp}(t)}{\overline{\varphi'^{\pm}(t)}}\right]=g^{\pm}(t) \tag{9}$$

where with (+) is marked the boundary value of the function for z $\in S^+$, and with (-) is marked the boundary value of the function for z $\in S^-$. The regions S^+ and S^- are determined by the following respective inequalities $|z|\langle 1$ and $|z|\rangle 1$ Constants a_0 and b_0 are expressed by the formulae (5).

$$a_0^2=\frac{\lambda+\mu}{\mu}\frac{2\mu(N_1+N_2)+N_1N_2+4\mu^2}{\lambda(N_1+N_2)-N_1N_2+4\mu(\lambda+\mu)},\quad b_0=\frac{(\lambda+2\mu)(N_1-N_2)e^{2i\alpha}}{\lambda(N_1+N_2)-N_1N_2+4\mu(\lambda+\mu)} \tag{10}$$

The main stress acting at the infinite point is marked with N_1 and N_2 and α is the angle which the main axis corresponding to N_1 forms with the co-ordinate axis 0X. The function g(t) is defined by the derivatives of the displacement vector components u'_θ and v'_θ toward the polar angle θ. The integrals in (9) are integrals of the type of Cauchy's. To these equations we will also add the expressing of $\psi(z)$ by the piece – wise analytical function $\varphi(z)$.

$$\psi'(z)=\frac{\bar{\varphi}(1/z)\varphi'(z)}{\varphi'^2(z)}+\frac{1}{z^2}\frac{\bar{\varphi}'(z)}{\varphi'(z)}-\frac{1}{z^2}\bar{\varphi}'^2(1/z), \tag{11}$$

In (11) $\bar{\varphi}(1/z)=\overline{\varphi(1/z)}$.

SOLUTION OF THE PROBLEM

We will solve the problem of determination the concentration coefficient of the stress when the elastic plane occupies S^- e.g. is an infinite plane with a circular hole and there's perfectly solid driving wheel soldered in, which radius R_2 is equal with R_1 of the disk and stresses. $\sigma_1^\infty=N_1,\ \sigma_2^\infty=N_2,\ \sigma_{12}^\infty=0,$ acting in the infinity.

We will accept $R_1=R_2=1$ for convenience in calculating. The functional equation (8) for the so formulated problem takes the descriptions:

$$\frac{\mu}{\lambda+\mu}\varphi'^2(z)-\frac{1}{2\pi i}\int_L\frac{\varphi^-(t)dt}{\varphi'^-(t)(t-z)^2}=\frac{\mu a_o^2}{\lambda+\mu}+\frac{\overline{b_0}}{z^2}, \text{ if } |z|\rangle 1 \quad (12)$$

We will determine the coefficient of stress concentration at the points z=$\pm$1. If we solve the problem concerning sufficiently limited area of these points then equation (12), will an acceptable approach to accuracy, can be substituted by the following approximate analogue.

$$\frac{\mu}{\lambda+\mu}\varphi'^2(z)-\frac{1}{2\pi i}\int_L\frac{\varphi^-(t)dt}{\varphi'^-(t)(t-z)^2}=\frac{\mu a_o^2}{\lambda+\mu}+\bar{b}_0, \quad \text{if } |z|\rangle 1 \quad (13)$$

The smaller radius of the studied areas the more precise substitution. Equation (13) has a precise solution

$$\varphi^-(z)=\sqrt{a_0^2-\frac{\lambda+\mu}{\mu}\overline{b_0}}.z, \qquad \text{if } |z|\rangle 1 \quad (14)$$

For the piece – wise analytical function $\varphi(z)$, after solving the nonlinear boundary problem (9) we get

$$\varphi(z)=\begin{cases}\varphi^+(z)=\left[\frac{1}{2A}+\frac{1}{2}\sqrt{\frac{1}{\overline{\overline{A^2}}}+4\left[\frac{\lambda+2\mu}{\lambda+\mu}-\frac{\mu}{\lambda+\mu}\lambda^2-\frac{A}{\overline{\overline{A}}}\right]}\right].z\\ \varphi^-(z)=\sqrt{a_0^2+\frac{\lambda+\mu}{\mu}\bar{b}_0}.z\end{cases} \quad (15)$$

Where $\quad A=\sqrt{a_0^2+\frac{\lambda+\mu}{\mu}\bar{b}_0} \quad (16)$

After substituting (15) in (11) and solving the problem for $N_1=0$, $N_2=N$ we will get for $\varphi(z)$ and $\psi(z)$ concerning $|z|\rangle 1$.

$$\varphi(z)=\frac{(\lambda+\mu)(4\mu^2-\lambda N)}{\mu[\lambda N+4\mu(\lambda+\mu)]}z, \qquad \psi(z)=\frac{\lambda(\lambda+2\mu)N}{(\lambda+\mu)[\lambda N+4\mu(\lambda+\mu)]}\frac{1}{z}. \quad (17)$$

These results and the equations (3) - (7) give the solution to the studied problem. Concretely, for the contour stress $\overline{\theta\theta}$ (in point coordinates) we get

$$\overline{\theta\theta}=\frac{2\lambda^2 N}{\lambda N-4\mu(\lambda+\mu)}=-\frac{2\lambda^2 N}{4\mu(\lambda+\mu)}\left[1-\frac{\lambda N}{4\mu(\lambda+\mu)}\right]^{-1} \quad (18)$$

We got a precise formula expressing $\overline{\theta\theta}$ Value at the point z=$\pm$1 . As it's known [5] analogous problem from the classical elastic theory

$$\overline{\theta\theta_L}=\frac{N}{2}\left[1-\frac{N(\mu-\lambda)}{(\lambda+\mu)(\lambda+3\mu)}\right] \quad (19)$$

If we set $\beta=\frac{\lambda}{\mu}$ and $\gamma=\frac{N}{\mu}$, then for the different values of β and γ for the coefficient of stress concentration

$$K=\frac{\overline{\theta\theta}}{N} \quad \text{and} \quad K_L \quad (20)$$

We get

…/…	0.2	0.4	0.6	0.8	1	Lin. Th.
0.25	- 0.0255	- 0.0260	- 0. 0268	- 0.0271	- 0.0284	0.0231
1	- o.2564	- 0.2632	- 0.2703	- 0.2778	- 0.2941	0
1.5	- 0.4663	- 0.4787	- 0.4945	- 0.5113	- 0.5488	-0.0333

SUMMARY

From the data in the schedule it's seen that the deflection from the classical solution in approximately title and that's because of the contour invariability in the process of deformation.

Following the described method we can find out the value of the coefficient of stress concentration at an arbitrary point of the contour of the area and we can get the full picture of the distribution of stress concentrations.

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