A Rothe-immersed Interface Method for an Elliptic-parabolic Interface Problem

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Absract. *This paper deals with the construction and theoretical analysis of a Rothe's FE-IIM for a model elliptic-parabolic problem. Numerical experiments are also discussed.*

Key words: *elliptic-parabolic problems, interface problems, FEM.*

INTRODUCTION

This paper concerns what we term a parabolic-elliptic interface problem. Two – dimensional problems are investigated in many papers [1-4]. They arise in the study of two-dimensional eddy currents [2], in the quasistationary two-dimensional magnetic fields [1], surface measurements [3], etc. In the present paper we shall concentrate on the following simplified model from the biochemical reactor theory [4]:

$$-\mu_E \frac{\partial^2 c}{\partial x^2} + f(x)c = g_1(x,y) \text{ in } \Omega_E = (0,\xi)\times(0,1), \tag{1}$$

$$v\frac{\partial c}{\partial y} - \mu_P \frac{\partial^2 c}{\partial x^2} = g_2(x,y) \text{ in } \Omega_P = (\xi,1)\times(0,1), \tag{2}$$

with boundary conditions

$$c(x,0) = c_0(x) \quad , \quad \xi < x < 1, \tag{3}$$

$$c(0,y) = g_3(y); c(1,y) = g_4(y); \; , \quad 0 < y < 1, \tag{4}$$

and interface conditions

$$[c]_{x=\xi} = c(\xi+,y) - c(\xi-,y) = 0, \tag{5}$$

$$[\mu\frac{\partial c}{\partial x}]_{x=\xi} = \mu_E \frac{\partial c}{\partial x}(\xi+,y) - \mu_P \frac{\partial c}{\partial x}(\xi-,y) = 0. \tag{6}$$

In (1), (2), μ_E, μ_P, are given positive constants.

The problem is elliptic in Ω_E and parabolic in Ω_P .

Let introduce the Sobolev space $H^1(0,1)$ and the bilinear form:

$$a(c,\varphi) = \int_0^\xi \frac{\partial c}{\partial x}\frac{d\varphi}{dx}dx + \frac{\mu_P}{\mu_E}\int_\xi^1 \frac{\partial c}{\partial x}\frac{d\varphi}{dx}dx, \quad \varphi \in H^1(0,1). \tag{7}$$

Let denote by $(,), (,)_E, (,)_P$ the inner products respectively in $L^2(0,1)$, $L^2(0,\xi)$, $L^2(\xi,1)$.

The equation (1) and (2) could be written in the equivalent form:

$$(v\frac{\partial c}{\partial y},\varphi) + \mu_E a(c,\varphi) + (f(x)c,\varphi) = 0 \quad \forall\varphi \in H^1(0,1). \tag{8}$$

Using energy methods in [6], one can prove that, if $f \in L^2(\Omega_E), g_1, g_2 \in L^2(\Omega_E \cup \Omega_P)$ then the problem (1)-(6) has unique solution $c \in L^2(0,1;\ H^1(0,1))$ and $\partial c/\partial x \in L^2(0,1;\ H^{-1}(0,1))$.

In Rothe's method [6] we apply an *y*–semidiscretization to approximate the parabolic part (1) of the problem by a finite sequence of elliptic interface boundary value problems. In Section 2 on each *y* – level we solve the corresponding elliptic problem by the FE-IIM, see [5]. Numerical experiments are discussed in the last section.

Rothe's FE-IIM

Let us divide the interval [0,1] by an equidistant mesh of step size $\tau = 1/M$. Let $z_m(x)$ denote the computed approximation of $c(x, y_m)$ at each y- level $y_m = m\tau,\ m = 0,1,\ldots,M$. These approximations are defined iteratively by

$$-\mu_E z_m''(x) + f(x) z_m = g_1(x, y_m), \quad x \in (0, \xi), \tag{9}$$

$$\nu \frac{z_m(x) - z_{m-1}(x)}{\tau} - \mu_P z_m''(x) = g_2(x, y_m), \quad x \in (\xi, 1) \tag{10}$$

$$z_0(x) = c_0(x),\ x \in (\xi, 1). \tag{11}$$

with boundary conditions

$$c_m(0, y_m) = g_3(y_m),\ c_m(1, y_m) = g_4(y_m), \tag{12}$$

and interface conditions

$$[z_m]_{x=\xi} = c(\xi+, y_m) - c(\xi-, y_m) = 0, \tag{13}$$

$$[\mu z']_{x=\xi} = 0\,. \tag{14}$$

This approximate solution can be extended from its values at the grid points y_j to all $y \in [0,1]$ by setting

$$c^\tau(x, y) = z_{m-1}(x) + \frac{y - y_{m-1}}{\tau}\left[z_m(x) - z_{m-1}(x)\right] \quad on \quad [y_{j-1}, y_j] \quad for \quad j = 1,\ldots,M.$$

We rewrite the equations (9) and (10) in the form

$$\left(\beta_m(x) z_m''\right) + q_m(x) z_m = r_m(x), \tag{15}$$

$$\beta_m(x) = \begin{cases} -\mu_E, \\ -\mu_P, \end{cases} \qquad q_m(x) = \begin{cases} f(x), \\ \dfrac{\nu}{\tau}, \end{cases} \qquad r_m(x) = \begin{cases} g_1(x, y_m), \\ g_2(x, y_m) + \dfrac{\nu}{\tau} z_{m-1}(x) \end{cases}.$$

Now we are in position to apply to the boundary value problem (15), (12), (13), (14) the FE-IIM proposed in [5]. We use an uniform grid $x_i = ih,\ i = 0,\ldots,N$ with $x_0 = 0, x_N = 1$ and $h = 1/N$. The standard linear basis function satisfies:

$$\phi_i = \begin{cases} 1, & if \quad i = k \\ 0, & \quad otherwise. \end{cases}$$

The numerical solution $z_m^h(x) = \sum_{i=0}^{N} c_i \phi_i(x)$, (with unknowns c_i) is a combination of the special basis, see [5] . If $x_j < \xi < x_{j+1}$, then ϕ_j and ϕ_{j+1} need to be changed to satisfy the second jump conditions (13), (14) which implies :

$$\phi_j(x) = \begin{cases} 0, & 0 \le x < x_{j-1} \\ \dfrac{x - x_{j-1}}{h}, & x_{j-1} \le x < x_j \\ \dfrac{x_j - x}{k}, & x_j \le x < \xi \\ \dfrac{\rho(x_{j+1} - x)}{k}, & \xi \le x < x_{j+1} \\ 0 & x_{j+1} \le x \le 1 \end{cases} \qquad \phi_{j+1}(x) = \begin{cases} 0, & 0 \le x < x_j \\ \dfrac{x - x_j}{k}, & x_j \le x < \xi \\ \dfrac{\rho(x - x_{j+1})}{k} + 1, & \xi \le x < x_{j+1} \\ \dfrac{x_{j+2} - x}{h}, & x_{j+1} \le x < x_{j+2} \\ 0 & x_{j+2} \le x \le 1 \end{cases}$$

where: $\rho = \dfrac{\nu\mu_E}{\mu_P}, \quad k = h - \dfrac{\nu\mu_E - \mu_P}{\nu\mu_E}(\xi - x_j)$.

Combining results from [5,6] one can prove the following theorem.

Theorem 1. Suppose that the assumptions for the weak form (7), (8) of problem (1)-(4) are fulfilled. Then for the Rothe's FE-IIM solution c_h^τ the estimate holds

$$\left\| c_h^\tau - c \right\| \le C\left(\tau + h^2\right), \tag{16}$$

where the constant C is independent of τ, h.

NUMERICAL EXPERIMENTS

For the numerical experiments we consider the following test problem:

$$-\mu_E \frac{\partial^2 c}{\partial x^2} + fc = g_1(x, y), \quad \text{in} \quad \Omega_E = (0, \frac{\pi}{6}) \times (0,1),$$

$$\frac{\partial c}{\partial y} - \mu_P \frac{\partial^2 c}{\partial x^2} = g_2(x, y), \quad \text{in} \quad \Omega_P = (\frac{\pi}{6}, 1) \times (0,1),$$

where we chose $\mu_E = 1, \quad \mu_P = \sqrt{3}, \quad f = 1, \; g_1(x, y) = g_2(x, y) = 0$, and an exact solution

$$c(x, y) = \begin{cases} \cos(\pi/6)\exp(\pi/6 - \sqrt{3}y - x), & 0 \le x \le \pi/6, 0 \le y \le 1 \\ \cos(x)\exp(\sqrt{3}y) & 0 \le x \le \pi/6, 0 \le y \le 1 \end{cases}. \tag{17}$$

The boundary conditions $g_3(y), g_4(y)$, and $c_0(x)$ are founded from the exact solution (17).

We examine two types of basis functions. First case: The interface ξ is a grid point, i.e. we divide $(0, \xi)$ in N_1 regular subintervals with mesh size $h_1 = \xi / N_1$ and $(\xi, 1)$ in N_2.

The numerical experiments are presented in Table 1, where $\| er \|_\infty = \max(| c(x_i, y_i) - c_h^\tau(i, j) |)$ is the error in maximum norm, ratio is $\text{ratio} = \dfrac{\| er^M_{N_1,N_2} \|_\infty}{\| er^{4M}_{2N_1,2N_2} \|_\infty}$, m is the rate of convergence, calculated by the formula $m = \log_2(ratio)$.

Ratio near 4 corresponds to second order of accuracy of the method in space direction x.

In Figure 1 the numerical solution for $N_1 = 40, N_2 = 40, M = 640$ is presented. Also, in Figure2 the error in maximum norm is depicted.

Second case: The interface ξ is not a grid point i.e. we divide (0,1) N regular intervals. The numerical experiments are given in Table 2. The results show again the second order of the method in depends of x variable. The error is bigger on the interface $x = \pi/6, \; y \in (0,1)$. In figure 3 the error in maximum norm if the interface $x = \xi$ is not a grid point, is presented.

Table1. Mesh refinement analysis for the case, when the interface is a grid point.

N1	N2	M	$\| er \|_\infty$	*ratio*	*m*
5	5	10	0.0036	-	-
10	10	40	9.8000e-004	3.6735	1.8771
20	20	160	2.5338e-004	3.8677	1.9515
40	40	640	6.3967e-005	3.9611	1.9859
80	80	2560	1.6030e-005	3.9905	1.9966

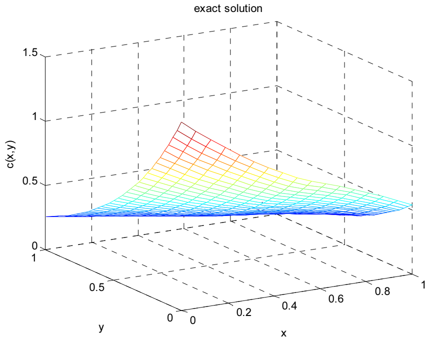

Figure1. The exact solution

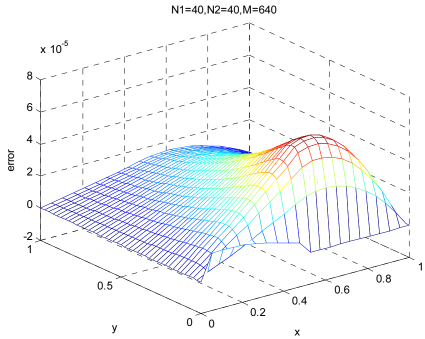

Figure 2. The error of the numerical solution in maximum norm, when the interface is a grid point.

Table 2. Mesh refinement analysis for the case, when the interface is not a grid point.

N	M	$\| c \|_\infty$	*ratio*	*m*
20	1000	0.0452	-	-
40	1000	0.0098	4.6122	2.2055
80	1000	0.0022	4.4545	2.1553
160	1000	5.3602e-004	4.1043	2.0371
320	1000	1.6400e-004	3.2684	1.7086

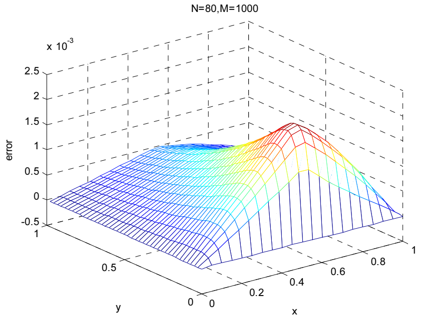

Figure3. The error of the numerical solution in maximum norm, when the interface is not a grid point.

CONCLUSION

The problem treated here is a simplification in on-space dimension of 2D-curve and 3D-surface interface problems occurring in physics and engineering. We have employed the Rothe method in combination with the FE-IIM. The experiment show that this method is very accurate for 1D problems and very promising for 2D. Progress has been made in the 2D-case , but further theoretical work is needed.

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