Discrete Dynamical Systems Generated by the Composition of Finite Mealy Automata

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Abstract: Discrete dynamical systems generated by the composition of finite Mealy automata: This paper considers the solutions of the following questions: the construction of the composition of random number of finite Mealy automata; the construction of inverse Mealy automaton and the solution of linear automaton equations; the development of suitable algorithms and.

Key words: discrete dynamical systems; Mealy automaton; composition; inverse automaton; linear automaton equation.

INTRODUCTION
Let us recall some fundamental definitions, modifying some of them, and introducing certain special notations and suitable general replacements[1],[2],[3],[4].

Definition 1.
A finite automaton M is a 6-tuple \( M = (Q, V, \delta, \lambda, q_0) \), where \( Q = \{q_0, q_1, \ldots, q_m\} \) is finite set of control states; \( V \) - starting and exit alphabet; \( \delta: Q \times V \rightarrow Q \) - state transition function; \( \lambda: Q \times V \rightarrow V_2 \) - outset function; \( q_0 \in Q \) is the initial state of the finite control (when \( V_1 \) coincides with \( V_2 \) - Mealy automaton).

Remark: The function \( \lfloor x \rfloor \) is the integer part of \( x \).

Now lets \( M_1 = (Q', V, \delta_1, \lambda_1, q_0') \), \( M_2 = (Q'', V, \delta_2, \lambda_2, q_0'') \) are Mealy automata.

Definition 2.
The composition, \( M = (Q, V, \delta, \lambda, q_0) \) is called composition of \( M_1 \) and \( M_2 \) and denotes
\[
M = M_2 \circ M_1,
\]
\[
\delta(q, x) = m_2(\delta_1(q', x) - 1) + \delta_2(q'', \lambda_1(q', x)),
\]
\[
\lambda(q, x) = \lambda_2(q'', \lambda_1(q', x)),
\]
as \( q' = \left\lfloor \frac{q + m_2 - 1}{m_2} \right\rfloor \), \( q'' = q - \left\lfloor \frac{q - 1}{m_2} \right\rfloor m_2 \).

An example to make it clear is done:
Example 1.
Let the finite automata \( M_1 \) and \( M_2 \) are given as follows:
\( M_1 = (Q', V, \delta_1, \lambda_1, q_0') \), where \( V = \{a, b, c\}; Q' = \{1, 2, 3, 4\}, q_0' = 1 \) and its functions \( \delta_1 \) and \( \lambda_1 \) are given in fig.1.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>a</td>
<td>2,b</td>
<td>3,c</td>
<td>4,b</td>
<td>1,a</td>
</tr>
<tr>
<td>b</td>
<td>4,c</td>
<td>1,a</td>
<td>2,c</td>
<td>3,b</td>
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<tr>
<td>c</td>
<td>3,a</td>
<td>4,b</td>
<td>1,a</td>
<td>2,c</td>
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</table>

fig. 1
$M_2 = \left( Q'', V, V, \delta_2, \lambda_2, q_0 '' \right)$, where $V = \{a, b, c\}; \; Q'' = \{1, 2, 3, 4\}, \; q_0 '' = 1$ and its functions $\delta_2$ and $\lambda_2$ are given in fig. 2.

<table>
<thead>
<tr>
<th>$M_2$</th>
<th>1</th>
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<tbody>
<tr>
<td>a</td>
<td>4b</td>
<td>3c</td>
<td>2a</td>
<td>2b</td>
</tr>
<tr>
<td>b</td>
<td>2c</td>
<td>4a</td>
<td>1c</td>
<td>3a</td>
</tr>
<tr>
<td>c</td>
<td>3a</td>
<td>3b</td>
<td>4b</td>
<td>1c</td>
</tr>
</tbody>
</table>

fig. 2

After some calculation (using program 2) of the functions $\delta$ and $\lambda$ according to formulae (1) and (2) the automaton $M = (Q, V, V, \delta, \lambda, q_0)$ can be defined as follows:

$V = \{a, b, c\}; \; Q = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}, \; q_0 = 1$ and its functions $\delta$ and $\lambda$ are given in fig. 3. (Calculations are done using suitable software developed by the author executing formulae (1) and (2)).

<table>
<thead>
<tr>
<th>$M$</th>
<th>1</th>
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<th>5</th>
<th>6</th>
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<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>6c</td>
<td>8a</td>
<td>5c</td>
<td>5a</td>
<td>11a</td>
<td>11b</td>
<td>12b</td>
<td>9c</td>
<td>14c</td>
<td>16a</td>
<td>13c</td>
<td>15a</td>
<td>4b</td>
<td>1c</td>
<td>2a</td>
<td>2b</td>
</tr>
<tr>
<td>b</td>
<td>15a</td>
<td>15b</td>
<td>16b</td>
<td>13c</td>
<td>4b</td>
<td>1c</td>
<td>2a</td>
<td>2b</td>
<td>7a</td>
<td>8b</td>
<td>5c</td>
<td>10c</td>
<td>12a</td>
<td>9c</td>
<td>11a</td>
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<tr>
<td>c</td>
<td>12b</td>
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<td>16a</td>
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<td>4b</td>
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<td>2b</td>
<td>7a</td>
<td>7b</td>
<td>8b</td>
<td>5c</td>
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</table>

fig. 3

For the word $\alpha_1 = bacac$ the two consequences with the values of $\delta_1, \lambda_1, \delta_2, \lambda_2$ and $\delta, \lambda$ can be found as follows:

$\delta_1(1, b) = 4; \; \lambda_1(1, b) = c; \; \delta_1(4, a) = 1; \; \lambda_1(4, a) = a; \; \delta_1(1, c) = 3; \; \lambda_1(1, c) = a; \; \delta_1(3, a) = 4; \; \lambda_1(3, a) = b; \; \lambda_1(4, c) = c; \; \Rightarrow M_1(\alpha_1) = caabc$

$\delta_2(1, c) = 3; \; \lambda_2(1, c) = a; \; \delta_2(3, a) = 2; \; \lambda_2(3, a) = a; \; \delta_2(2, a) = 1; \; \lambda_2(2, a) = c; \; \delta_2(1, b) = 2; \; \lambda_2(1, b) = c; \; \lambda_2(2, c) = b; \; \Rightarrow M_2(M_1(\alpha_1)) = aaccb$

$\delta(1, b) = 15; \; \lambda(1, b) = a; \; \delta(15, a) = 2; \; \lambda(15, a) = a; \; \delta(2, c) = 9; \; \lambda(2, c) = c; \; \delta(9, a) = 14; \; \lambda(9, a) = c; \; \lambda(14, c) = b; \; \Rightarrow M(\alpha_1) = aaccb = M_2(M_1(\alpha_1))$
An investigation on the word $\alpha_2 = aacbbbac$ follows:
\[
\begin{align*}
\delta_1(1,a) & = 2; \quad \lambda_1(1,a) = b; \\
\delta_1(2,a) & = 3; \quad \lambda_1(2,a) = c; \\
\delta_1(3,a) & = 1; \quad \lambda_1(3,a) = a; \\
\delta_1(1,c) & = 3; \quad \lambda_1(1,c) = a; \\
\delta_1(3,b) & = 2; \quad \lambda_1(3,b) = c; \\
\delta_1(2,b) & = 1; \quad \lambda_1(2,b) = a; \\
\delta_1(1,a) & = 2; \quad \lambda_1(1,a) = b; \\
\lambda_1(2,c) & = b; \\
\Rightarrow M_1(\alpha_2) & = bcaacabb \\
\delta_2(1,b) & = 2; \quad \lambda_2(1,b) = c; \\
\delta_2(2,c) & = 3; \quad \lambda_2(2,c) = b; \\
\delta_2(3,a) & = 2; \quad \lambda_2(3,a) = a; \\
\delta_2(2,a) & = 1; \quad \lambda_2(2,a) = c; \\
\delta_2(1,c) & = 3; \quad \lambda_2(1,c) = a; \\
\delta_2(3,a) & = 2; \quad \lambda_2(3,a) = a; \\
\delta_2(2,b) & = 4; \quad \lambda_2(2,b) = a; \\
\lambda_2(4,b) & = a; \\
\Rightarrow M_2(M_1(\alpha_2)) & = cbacaaaa \\
\delta(1,a) & = 6; \quad \lambda(1,a) = c; \\
\delta(6,a) & = 11; \quad \lambda(6,a) = b; \\
\delta(11,c) & = 2; \quad \lambda(11,c) = a; \\
\delta(2,c) & = 9; \quad \lambda(2,a) = c; \\
\delta(9,b) & = 7; \quad \lambda(9,b) = a; \\
\delta(7,b) & = 2; \quad \lambda(7,b) = a; \\
\delta(2,a) & = 8; \quad \lambda(2,a) = a; \\
\lambda(8,c) & = a; \\
\end{align*}
\]
Then $M(\alpha_2) = cbacaaaa = M_2(M_1(\alpha_2))$

This example gives us grounds to formulate and prove the following theorem:

**Theorem 1.** The composition $M = M_2 \circ M_1$ of two Mealy automata generates the automaton map $\varphi = \varphi_2 \circ \varphi_1$.

**Proof**

Let $M = (Q,V,\delta,\lambda,q_0)$ be the Mealy automaton, constructed above, $\alpha \in V^*$, $\alpha = a_1a_2a_3...a_k$,
\[
\begin{align*}
\delta_1(q_0',a_1) & = q_1'; \quad \delta_1(q_{i-1}',a_{i}) = q_i; \quad \lambda_1(q_0',a_1) = b_1; \quad \lambda_1(q_{i-1}',a_{i}) = b_i, \\
\delta_2(q_0'',b_1) & = q_1''; \quad \delta_2(q_{i-1}'',b_i) = q_i; \quad \lambda_2(q_0'',b_1) = c_1; \quad \lambda_2(q_{i-1}'',b_i) = c_i, \\
k & = 1,\ldots,s.
\end{align*}
\]

Thus $M_1(\alpha) = b_1b_2b_3...b_k$, $M_2(M_1(\alpha)) = c_1c_2c_3...c_k$. 

- 28 -
An investigation on \( a_i \) and \( a_{ik} \) (for any \( k = 2,\ldots,s \)) follows.

\[
\begin{align*}
\delta_1(q_0', a_i) &= q_1', \\
\lambda_1(q_0', a_i) &= b_1', \\
\delta_2(q_0'', b_i) &= q_2'', \\
\lambda_2(q_0'', b_i) &= c_2',
\end{align*}
\]

which means \( M_2(M_1(a_i)) = c_i \).

\[
\begin{align*}
\delta(q_0', a_i) &= m_2(\delta(q_0', a_i) - 1) + \delta_2(q_0'', \lambda_i(q_0', a_i)) = m_2(q_0' - 1) + q_0'' \\
\lambda(q_0', a_i) &= \lambda_2(q_0'', \lambda_i(q_0', a_i)) = \lambda_2(q_0'', b_i) = c_i \\
\end{align*}
\]

and so \( M(a_i) = c_i = M_2(M_1(a_i)) \).

The same way can be obtained the result for \( a_{ik} \):

\[
\begin{align*}
\delta(q_{ik-1}', a_{ik}) &= q_{ik}'', \\
\lambda(q_{ik}', a_{ik}) &= b_{ik}'', \\
\delta_2(q_{ik-1}''', b_{ik}) &= q_{ik}''', \\
\lambda_2(q_{ik-1}''', b_{ik}) &= c_{ik}'',
\end{align*}
\]

which means \( M_2(M_1(a_{ik})) = c_{ik} \).

\[
\begin{align*}
\delta(q_{ik-1}, a_{ik}) &= m_2(\delta(q_{ik-1}', a_{ik}) - 1) + \delta_2(q_{ik-1}'', \lambda_{ik}(q_{ik-1}', a_{ik})) = m_2(q_{ik} - 1) + q_{ik}'' \\
\lambda(q_{ik-1}'', a_{ik}) &= \lambda_2(q_{ik-1}''', \lambda_{ik}(q_{ik-1}', a_{ik})) = \lambda_2(q_{ik-1}''', b_{ik}) = c_{ik}.
\end{align*}
\]

Therefore \( M(a_{ik}) = c_{ik} = M_2(M_1(a_{ik})) \) and hence \( M(\alpha) = M_2(M_1(\alpha)) \) so in conclusion \( M = M_2 \circ M_1 \).

**LITERATURE**


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