An algorithm for solving a Sylvester quaternion equation

Georgi Georgiev, Ivan Ivanov, Milena Mihaylova, Tsvetelina Dinkova

Abstract: An algorithm for solving a Sylvester quaternion equation: *The paper deals with the solutions of a linear equation of one quaternion unknown. We find solutions of a Sylvester quaternion equation by a reduction to a simple matrix equation. Using computer algebra systems MATHEMATICA and MATLAB we obtain programs for symbolic and numerical presentation of these solutions. Important particular cases and numerical examples are also considered.*

Key words: *Quaternion algebra, Sylvester equation , Commuting quaternions..*

INTRODUCTION

The quaternion algebra is associative but non-commutative . Therefore there are different kinds of linear equations of one quaternion unknown. These equations and their solutions in separate cases are discussed in [1]. The aim of this paper is to give the common method for solving a Sylvester quaternion equation.

THE QUATERNION ALGEBRA

We consider the algebra $\mathbf{H}$ of quaternions defined as a four-dimensional vector space over $\mathbf{R}$ with basic elements e, i, j, k i.e.
$\mathbf{H} = \{a_1 e + a_2 i + a_3 j + a_4 k \,/\, a_i \in \mathbf{R}, i = 1 \div 4\}$ and
$i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. If $a, b \in \mathbf{H}$,
$a = a_1 e + a_2 i + a_3 j + a_4 k$ and $b = b_1 e + b_2 i + b_3 j + b_4 k$, then
$a + b = (a_1 + b_1)e + (a_2 + b_2)i + (a_3 + b_3)j + (a_4 + b_4)k$ and
$ab = (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)e + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)i +$
$+ (a_1 b_3 + a_3 b_1 + a_4 b_2 - a_2 b_4)j + (a_1 b_4 + a_4 b_1 + a_2 b_3 - a_3 b_2)k$.

Hence, $\mathbf{H}$ is associative non-commutative algebra with the unit element e and $\mathbf{R}$ is a center of $\mathbf{H}$ Quaternions can be also written as four-dimensional vectors $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbf{R}^4$ with above rules for addition and multiplication. The norm of the quaternion $a = a_1 e + a_2 i + a_3 j + a_4 k$ is the non-negative real number $\| a \| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$.

Let us consider the space of pure imaginary quaternions $\operatorname{Im}\mathbf{H} = \{a_2 i + a_3 j + a_4 k \,/\, a_2, a_3, a_4 \in \mathbf{R}\}$ which is a three-dimensional vector subspace of $\mathbf{H}$. Then every quaternion $a = a_1 e + a_2 i + a_3 j + a_4 k$ can be represented uniquely in the form $a = \operatorname{Re}(a)e + \operatorname{Im}(a)$, where $\operatorname{Re}(a) = a_1$ is called a scalar part of a and $\operatorname{Im}(a) = a_2 i + a_3 j + a_4 k$ is called a vector part of a. More details about quaternions can be found in [2] and [3].

A SYLVESTER QUATERNION EQUATION

Since $\mathbf{H}$ is non-commutative algebra we may consider the following equation of one quaternion variable x

$$ax + xb = c, \tag{1}$$

where $a, b, c \in \mathbf{H}$ and $ab \neq 0$. The above equation (1) is called *a Sylvester quaternion equation*. In the whole article we assume that $a, b \in \mathbf{H} \setminus \mathbf{R}$. It is shown in [1] that the solution of the equation (1) is unique if and only if either $\operatorname{Re}(a) \neq -\operatorname{Re}(b)$ or

$\| a \| \neq \| b \|$. Moreover, this unique solution is obtained in an explicit form. There exist correspondences between the quaternion algebra $\mathbf{H}$ and special kinds of 4×4 real matrices (see [1] and [3]). Following [1] we consider the mappings $i_1 : \mathbf{H} \to \mathbf{R}^{4\times 4}$ and $i_2 : \mathbf{H} \to \mathbf{R}^{4\times 4}$ given by

$$i_1(a) = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \quad \text{and} \quad i_2(a) = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}.$$

Then, the Sylvester equation (1) is equivalent to the matrix equation

$$\mathbf{M}\,\mathbf{x}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}},$$

where $\mathbf{M} = i_1(a) + i_2(b)$ is a 4×4 real matrix, $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{c} = (c_1, c_2, c_3, c_4)$ are the vectors corresponding to the quaternions $x \in \mathbf{H}$ and $c \in \mathbf{H}$, respectively (see [1], Theorem 3.2)..

Proposition 1. *The rank of the matrix* $\mathbf{M}$ *is even.*

Proof: By direct calculations we obtain the eigenvalues of the matrix $\mathbf{M}$

$$\lambda_{1,2} = a_1 + b_1 \pm \sqrt{-a_2^2 - a_3^2 - a_4^2 - b_2^2 - b_3^2 - b_4^2 - 2\sqrt{(a_2^2 + a_3^2 + a_4^2)\,(b_2^2 + b_3^2 + b_4^2)}}$$

$$\lambda_{3,4} = a_1 + b_1 \pm \sqrt{-a_2^2 - a_3^2 - a_4^2 - b_2^2 - b_3^2 - b_4^2 + 2\sqrt{(a_2^2 + a_3^2 + a_4^2)\,(b_2^2 + b_3^2 + b_4^2)}}\,.$$

From here it follows that the four eigenvalues are different and nonzero, or equivalently, $\operatorname{rank}(\mathbf{M}) = \mathbf{4}$, if $a_1 \neq -b_1$ or $\| \operatorname{Im}(a) \| \neq \| \operatorname{Im}(b) \|$. When $a_1 = -b_1$ and $\| \operatorname{Im}(a) \| = \| \operatorname{Im}(b) \|$, we observe that $\lambda_1 = -\lambda_2 \neq 0$, $\lambda_3 = \lambda_4 = 0$. Therefore, we have $\operatorname{rank}(\mathbf{M}) = \mathbf{2}$ in this case.

Let $\mathbf{S}$ be the vector space spanned by the columns of the matrix $\mathbf{M}$. This vector subspace of $\mathbf{R}^4$ can be represented as $\mathbf{S} = \langle \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4 \rangle$, where

$$\mathbf{m}_1 = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)^T \;, \quad \mathbf{m}_2 = (-a_2 - b_2, a_1 + b_1, a_4 - b_4, -a_3 + b_3)^T$$

$$\mathbf{m}_3 = (-a_3 - b_3, -a_4 + b_4, a_1 + b_1, a_2 - b_2)^T \;, \quad \mathbf{m}_4 = (-a_4 - b_4, a_3 - b_3, -a_2 + b_2, a_1 + b_1)^T$$

Proposition 2. *Let* $a, b, c, x \in \mathbf{H}$ *and* $ab \neq 0$. *Then Sylvester equation* $ax + xb = c$ *has:*

1) *a unique solution if and only if* $rank(\mathbf{M}) = \mathbf{4}$;
2) *an infinite number of solutions if and only if* $rank(\mathbf{M}) = \mathbf{2}$ *and* $\mathbf{c}^{\mathrm{T}} \in \mathbf{S}$;
3) *no solution if and only if* $rank(\mathbf{M}) = \mathbf{2}$ *and* $\mathbf{c}^{\mathrm{T}} \notin \mathbf{S}$.

Proof: 1) The equation $ax + xb = c$ has one solution whenever the matrix equation $\mathbf{M}\,\mathbf{x}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}}$ has one solution. That is equivalent to $\operatorname{rank}(\mathbf{M}) = \mathbf{4}$.

2) The equation $ax + xb = c$ has an infinite number of solutions when the system of linear equations corresponding to $\mathbf{M}\,\mathbf{x}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}}$ has an infinite number of solutions. This is possible if and only if $\operatorname{rank}(\mathbf{M}) < \mathbf{4}$ and $\operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\overline{\mathbf{M}})$, where $\overline{\mathbf{M}}$ is a 4×5 matrix containing $\mathbf{M}$ and an additional column $\mathbf{c}^{\mathrm{T}}$. Using Proposition 1 we conclude that $\operatorname{rank}(\mathbf{M}) = \mathbf{2}$, and $\mathbf{c}^{\mathrm{T}} \in \mathbf{S}$ in this case.

3) From the assertions 1) and 2) it follows that the equation $ax + xb = c$ has no solution when $\operatorname{rank}(\mathbf{M}) = \mathbf{2}$ and $\mathbf{c}^{\mathrm{T}} \notin \mathbf{S}$.

AN ALGORITHM

We propose a simple algorithm for determining the solutions of the Sylvester equation (1). This algorithm is based on Proposition 2 and it will be written down in mathematical style pseudo code.

If *rank(*$\mathbf{M}$*)=4* ***then*** *the unique solution of the equation (1) written in a vector form is* $\mathbf{x}^{\mathrm{T}} = \mathbf{M}^{-1}\,\mathbf{c}^{\mathrm{T}}$

else if *rank(*$\mathbf{M}$ *)* $\neq$ *rank(*$\overline{\mathbf{M}}$ *)* ***then*** *print "has no solution"*

else *Find the general solution of* $\mathbf{M}\,\mathbf{x}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}}$ *that depends on two parameters*

Realization of the algorithm with the computer algebra system MATLAB.

We suggest the file - function SQE with three input arguments. This function written as M – file with the same name can solve the Sylvester's equation many times.

```
function x=SQE(a,b,c)
I1=[a(1) -a(2) -a(3) -a(4);a(2) a(1) -a(4) a(3);a(3) a(4) a(1) -a(2);a(4) -a(3) a(2) a(1)];
I2=[b(1) -b(2) -b(3) -b(4);b(2) b(1) b(4) -b(3);b(3) -b(4) b(1) b(2);b(4) b(3) -b(2) b(1)];
M=I1+I2; M=sym(M);
if rank(M)==4
   X=M\c.';  x=X.';
elseif rank(M)~=rank([M c'])
   disp(' The equation has no solution ');
else Z=null(M); syms p q; X1=p*Z(:,1)+q*Z(:,2);
      X=X1+M\c.'; x=X.';
end end
```

Example 1: We will solve the Sylvester equation $ax + xb = c$ with known a, b, c:

a) $a = 5e + i + 7j - 2k$ $\quad b = e + 4i + 2j - 3k$ $\quad c = -20e - 9i + 29j - 26k$

b) $a = 4e + 2i + j + 3k$ $\quad b = -4e - 3i + j + 2k$ $\quad c = 15e - i + 17j + 5k$

c) $a = -3e + i + 7j - 6k$ $\quad b = 3e + 6i + j - 7k$ $\quad c = 11e + 5i + 6j + 4k$

d) $a = -e + 3i + 4j + 8k$ $\quad b = 2e - 3i + 5j + k$ $\quad c = 0$

e) $a = -2e + 5i + j + 4k$ $\quad b = 2e - 4i + 5j - k$ $\quad c = 0$

a)
```
>> a=[5 1 7 -2]; b=[1 4 2 -3]; c=[-20 -9 29 -26];
>> x=SQE(a,b,c)
x =
[  2, -1,  3, -2]
```

b)
```
>> a=[4 2 1 3]; b=[-4 -3 1 2]; c=[15 -1 17 5];
>> x=SQE(a,b,c)
Warning: System is rank deficient. Solution is not unique.
> In C:\MATLAB6p5\toolbox\symbolic\@sym\mldivide.m at line 38
  In C:\MATLAB6p5\work\SQE.m at line 10
x =
[      -p+1, 2*p+5*q+15,          p,          q]
```

c)
```
>> a=[-3 1 7 -6]; b=[3 6 1 -7]; c=[11 5 6 4];
>> x=SQE(a,b,c)
The equation has no solution.
??? One or more output arguments not assigned during call to 'sqe'.
```

d)
```
>> a=[-1 3 4 8]; b=[2 -3 5 1]; c=[0 0 0 0];
>> x=SQE(a,b,c)
x =
[0, 0, 0, 0]
```

e)
```
>> a=[-2 5 1 4]; b=[2 -4 5 -1]; c=[0 0 0 0];
>> x=SQE(a,b,c)
```

```
Warning: System is rank deficient. Solution is not unique.
> In C:\MATLAB6p5\toolbox\symbolic\@sym\mldivide.m at line 38
  In C:\MATLAB6p5\work\SQE.m at line 10
x =
[  4*p+5*q,  -3*p-6*q,        q,        p]
```

Example 2: Find all quaternions which commutate with a given quaternion $a = 13e - 21i + 5j - 8k$.

The required quaternion x and a commutate if and only if $ax = xa$. This is equivalent to $ax - xa = 0$. The problem is reduced to solving the Sylvester's equation. We will use file - function SQE.

```
>> a=[13 -21 5 -8]; b= -a; c=[0 0 0 0];
>> x=SQE(a,b,c)
Warning: System is rank deficient. Solution is not unique.
> In C:\MATLAB6p5\toolbox\symbolic\@sym\mldivide.m at line 38
  In C:\MATLAB6p5\work\SQE.m at line 10
x =
[       q,  -21/5*p,        p,  -8/5*p]
```

Consequently all quaternions which commutate with a are $x = qe - 21/5pi + pj - 8/5pk$, $p, q \in \mathbf{R}$.

Realization of the algorithm with the computer algebra system MATHEMATICA.

SQE [a_ ,b_ ,c_] :=

$$i_1(a) = \begin{pmatrix} a[[1]] & -a[[2]] & -a[[3]] & -a[[4]] \\ a[[2]] & a[[1]] & -a[[4]] & a[[3]] \\ a[[3]] & a[[4]] & a[[1]] & -a[[2]] \\ a[[4]] & -a[[3]] & a[[2]] & a[[1]] \end{pmatrix}; \; i_2(b) = \begin{pmatrix} b[[1]] & -b[[2]] & -b[[3]] & -b[[4]] \\ b[[2]] & b[[1]] & b[[4]] & -b[[3]] \\ b[[3]] & -b[[4]] & b[[1]] & b[[2]] \\ b[[4]] & b[[3]] & -b[[2]] & b[[1]] \end{pmatrix};$$

$M = i_1(a) + i_2(b)$; If [MatrixRank [M] == 4 , Inverse [M] . c , If [MatrixRank[MapThread[Append , { M , c }]] $\neq$ MatrixRank [M] , Print ["The equation ax+xb=c has not solution "] , $y = \{x_1, x_2, x_3, x_4\}$; Solve [{ M[[1]].y==c[[1]], M[[2]].y==c[[2]] ,M[[3]].y==c[[3]] ,M[[4]].y==c[[4]] }, $\{x_1, x_2, x_3, x_4\}$]]] ;

Some particular cases. We consider the following cases for the Sylvester equation $ax + xb = c$:

1) Let $a = \alpha e + i, \quad b = -\alpha e + i$ and $c = c_1 e + c_2 i$. Then

$$\mathbf{M} = i_1(a) + i_2(b) = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\mathbf{c}^{\mathrm{T}}$ is a linear combination of the first and the second column of $\mathbf{M}$:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\mu \\ 2\lambda \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda, \mu \in \mathbf{R} .$$

Finally, we calculate the general solution with MATHEMATICA

$$\mathbf{x} = \left(x_1 \to \frac{c_2}{2},\ x_2 \to \frac{-c_1}{2},\ p, q \right) \quad p, q \in \mathbf{R} .$$

2) Let $a = \alpha e + i, \quad b = -\alpha e + j$ and $c = c_1 e + c_2 i + c_2 j - c_1 k$. Then the rank of the matrix $\mathbf{M} = i_1(a) + i_2(b) = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ is equal to 2 . Moreover $\mathbf{c}^{\mathrm{T}}$ is a linear combination of the first and the second column of $\mathbf{M}$:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\mu \\ \lambda \\ \lambda \\ \mu \end{pmatrix} , \qquad \lambda, \mu \in \mathbf{R} .$$

The general solution is $\mathbf{x} = (x_1 \to c_2 + q,\ x_2 \to -c_1 - p,\ p,\ q) \quad p, q \in \mathbf{R}$.

3) Let $a = a_1 e + a_2 i + a_3 j + a_4 k$ and $a_4 \neq 0$. We will find the solutions of homogeneous equation $ax - xa = 0$. Since the rank of the matrix

$$\mathbf{M} = i_1(a) + i_2(-a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2a_4 & 2a_3 \\ 0 & 2a_4 & 0 & -2a_2 \\ 0 & -2a_3 & 2a_2 & 0 \end{pmatrix}$$

is equal to 2 all quaternions which commuting with a form a two parameter set $\mathbf{x} = \left(p, x_2 \to \frac{a_2 q}{a_4}, x_3 \to \frac{a_3 q}{a_4}, q \right) p, q \in \mathbf{R}$.

REFERENCES

[1] Janovská,D., G.Opfer. Linear equations in quaternionic variables. Mitt. Math. Ges. Hamburg, **27** (2008), 223-234.

[2] Quipers,J. *Quaternions and rotation sequences*. Princeton University Press, Princeton , 2002.

[3] Ward,J. *Quaternions and Cayley Numbers: Algebra and Applications*. Kluwer Academic Publishers, Dordrecht, 1997.

За контакти:

Доц. д-р Георги Христов Георгиев, ст. ас. Милена Николова Михайлова, ст. ас. Иван Славейков Иванов, ст. ас. Цветелина Лъчезарова Динкова, Катедра "Алгебра и геометрия", ФМИ, Шуменски университет "Епископ Константин Преславски", e-mail: g.georgiev@fmi.shu-bg.net; nicolova_m@abv.bg; slaveicov@abv.bg; cvetelina_d@abv.bg

Докладът е рецензиран.