A near-exact distribution and exact percentage points for testing independence with missing elements in the sample correlation matrix

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Abstract: This paper consider the likelihood ratio test for diagonality of the covariance matrix of a \( p \)-variate normal distribution, when the sample correlation matrix has missing elements. The exact null distribution of the test statistic is represented through Meijer G-functions. For small values of \( p \), percentage points for \( \alpha=0.05 \) and \( \alpha=0.01 \) are computed. For calculation of quantiles when \( p \) is large, a near-exact null distribution of the test statistic is derived in the form of a Generalized Near-Integer Gamma distribution.

Key words: Product of beta random variables, Likelihood ratio test, Percentage points, Meijer G-function, Near-exact distribution, Generalized Near-Integer Gamma distribution, Monte Carlo simulation

INTRODUCTION

The \( ([2/n]^n)^{-1} \)-th power of the likelihood ratio test statistic for independence of \( p \) multivariate normal distributed random variables, based on a sample of size \( n \), is the statistic (see [1])

\[
L = \det R,
\]

where \( R \) is the sample correlation matrix. Often in practice, the missing observations on some of the \( p \) random variables can lead to missing elements in the sample correlation matrix. For instance, if we have \( n_1 \) observations on the first \( p - 1 \) random variables and \( n_2 \) realizations of the variables with numbers from \( k + 1 \) to \( p \), then in the sample correlation matrix \( R = (r_{i,j}) \) the elements \( r_{i,p}, \ldots, r_{k,p} \) will be unidentified. Another reason for missing elements in the sample correlation matrix might be a loss during the keeping or the transportation to the researcher. To check the independence of \( p \) multivariate normal distributed random variables, when the elements \( r_{i,p}, \ldots, r_{k,p}, 1 \leq k \leq p - 2 \) of the sample correlation matrix \( R = (r_{i,j}) \) are missing, the likelihood ratio test is derived in [5]. Denoted by \( L_k \), the \( [2/(n-1)]^{-1} \)-th power of this test statistic is given below in (2). Under the null hypothesis that the \( p \) variables are independent, \( L_k \) is distributed as the product of \( p - 1 \) beta random variables, as it is shown in [6].

In this paper we give a representation of the null distribution of \( L_k \) in terms of Meijer G-functions. Using both the exact mathematical formulas with a commercial mathematical software and Monte Carlo simulations, some percentage points for small values of \( p \) are computed. For calculation of quantiles when \( p \) is large, a near-exact distribution for the null distribution of \( L_k \) is derived.

THE EXACT NULL DISTRIBUTION OF \( L_k \)

Let \( A \) be a real square matrix of order \( n \). Let \( \alpha \) and \( \beta \) be nonempty subsets of the set \( N_n = \{1, \ldots, n\} \). By \( A[\alpha, \beta] \) we denote the submatrix of \( A \), composed of the rows with numbers from \( \alpha \) and the columns with numbers from \( \beta \). When \( \beta = \alpha \), \( A[\alpha, \alpha] \) is denoted simply by \( A[\alpha] \).

The \( [2/(n-1)]^{-1} \)-th power of the likelihood ratio test statistic for independence of the \( p \) variables is shown in [5] to be

\[
L_k = \frac{\det R[k+1,\ldots,p]\det R[1,\ldots,p-1]}{\det R[k+1,\ldots,p-1]},
\]

(2)
where \( k \) is the number of missing elements in the last column of the sample correlation matrix and \( n \) is the total number of observations on the \( p \) random variables. The likelihood ratio is derived under the assumption that we do not hold the initial observations, and have only the sample correlation matrix, in which the elements \( r_{k,p}, \ldots, r_{k,p}, 1 \leq k \leq p-2 \) are missing.

From (2) It can be seen that when \( k = 0 \), i.e. there are no missing elements in the sample correlation matrix, \( L_k \) equals the usual test statistic (1) for independence of \( p \) random variables.

Subsequently, \( X \sim Beta(\alpha, \beta) \) denotes the classical beta random variable defined on \([0,1]\), with density \( f(x; \alpha, \beta) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha, \beta) \), with \( B(\alpha, \beta) \) being the beta function in \((\alpha, \beta)\).

In [6] it is shown that under the null hypothesis of independence of the \( p \) variables, \( L_k \) is distributed as a product of \( p-1 \) independent beta random variables

\[
L_k \cong \prod_{j=1}^{p-1} X_j, \quad (3)
\]

\( X_j \sim Beta\left(\frac{n-p+j}{2}, \frac{p-j}{2}\right), \quad j = 1, \ldots, k, \quad X_j \sim Beta\left(\frac{n-p+j-1}{2}, \frac{p-j}{2}\right), \quad j = k+1, \ldots, p-1. \)

The density \( g(u) \) of the product of \( p \) independently distributed beta random variables with parameters \((\alpha_j, \beta_j), \quad j = 1, \ldots, p\) can be expressed in terms of Meijer G-functions (see [4], p.51) as follows

\[
g(u) = \prod_{j=1}^{p} \Gamma(\alpha_j + \beta_j) \Gamma(\alpha_j) G_{p,p}^{p,0}\left[ \begin{array}{c} \alpha_j + \beta_j - 1, \quad j = 1, \ldots, p \\ \alpha_j - 1, \quad j = 1, \ldots, p \end{array} \right], \quad 0 < u < 1. \quad (4)
\]

From (3) and (4) we obtain a representation for the density of the null distribution of \( L_k \):

\[
f_{L_k}(x) = KG_{p-1,p-1}^{p-1,0}\left[ \begin{array}{c} n-3 \\ n-3 \\ n-3 \\ \ldots \\ n-3 \\ n-p-1 \\ n-p-k-2 \\ n-p-k-2 \\ \ldots \\ n-p-k-2 \\ n-4 \\ \ldots \\ n-4 \\ \ldots \\ n-4 \\ \ldots \\ n-4 \end{array} \right], \quad 0 < x < 1, \quad (5)
\]

with \( K = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-p+k}{2}\right)} \prod_{j=1}^{p-2} \Gamma\left(\frac{n-p+j}{2}\right). \)

The commercial mathematical softwares, like MAPLE and MATHEMATICA, are able to compute the Meijer G-functions. For calculation of the \( \alpha \)-quantile of the null distribution of \( L_k \), we need to solve the equation \( \int_0^u f_{L_k}(x)dx = \alpha \) with respect to \( u \). It still takes too much computer time compare to the calculation of quantiles using Monte Carlo simulation techniques.

**A NEAR-EXACT NULL DISTRIBUTION OF \( L_k \)**

A near-exact distribution theory for the most common likelihood ratio test statistics used in multivariate analysis is generalized in [3]. The near-exact distributions are much closer to the exact distributions than common asymptotic distributions are. They are known manageable distributions, from which quantiles and p-values may be easily computed.

To obtain a near-exact distribution for the null distribution of \( L_k \), where \( k \) is an arbitrary integer, \( 1 \leq k \leq p-2 \), we shall use an approach similar to those in [2] for the case \( k = 0 \).
Let \( W = -\log L_k \) and \( \phi_p(t) \) be the characteristic function of \( W \). Since for \( X \sim Beta(\alpha, \beta) \), we have \( E(X^s) = \frac{\Gamma(\alpha + s)\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + s)\Gamma(\alpha)} \), \( s > -\alpha \), using (3) we obtain that

\[
\phi_p(t) = E(e^{at}) = \prod_{j=1}^{\nu-1} E(X_j^{-it}) = \left( \prod_{j=1}^{\nu-1} \frac{\Gamma(\frac{n-p+j}{2} - it)\Gamma(\frac{n-1}{2})}{\Gamma\left(\frac{n-p+j}{2}\right)\Gamma\left(\frac{n-1}{2} - it\right)} \right) \left( \prod_{j=k+1}^{\nu} \frac{\Gamma\left(\frac{n-p+j-1}{2} - it\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-p+j-1}{2}\right)\Gamma\left(\frac{n-1}{2} - it\right)} \right),
\]

where \( i = (-1)^{1/2} \) is the imaginary unit. In [2], the second factor in (6) is written of the form

\[
\prod_{j=1}^{\nu-1} \frac{\Gamma\left(\frac{n-p+j-1}{2} - it\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-p+j}{2}\right)\Gamma\left(\frac{n-1}{2} - it\right)} = \prod_{j=1}^{\nu-1} \frac{\Gamma\left(\frac{n-2-j}{2} - it\right)\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-2-j}{2}\right)\Gamma\left(\frac{n-2}{2} - it\right)},
\]

where \( \lfloor x \rfloor \) denotes the largest integer not greater than \( x \). We are using the same approach as in [2] to deal with the complementary factor in (6) and derive \( \phi_p(t) \) of the form

\[
\phi_p(t) = \left( \prod_{j=1}^{\nu-1} \frac{\Gamma\left(\frac{n-2-j}{2} - it\right)\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-2-j}{2}\right)\Gamma\left(\frac{n-2}{2} - it\right)} \right)^{\frac{p-1}{2}} \prod_{j=1}^{\nu-1} \frac{\Gamma\left(\frac{n-2-j}{2} - it\right)\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-2-j}{2}\right)\Gamma\left(\frac{n-2}{2} - it\right)},
\]

where

\[
\delta = \begin{cases} 
-1, & \text{if } k \text{ is odd and } p \text{ is even} \\
1, & \text{if } k \text{ is odd and } p \text{ is odd} \\
0, & \text{if } k \text{ is even}
\end{cases}, \quad \epsilon_j = \begin{cases} 
1, & \text{if } p-k-j \text{ is positive and even} \\
0, & \text{otherwise}
\end{cases}
\]

Let us denote the first factor in the right hand side of (7) by \( \phi_p(t) \). Replacing in (7) \( \phi_p(t) \) with the function \( \mu^r (\mu - it)^r \), which is the characteristic function of a Gamma distributed random variable with parameters \( \mu \) and \( r \), we obtain the characteristic function of a near-exact distribution of \( W \). For

\[
\mu = \frac{m_1}{m_2 - m_1}, \quad r = \frac{m_2^2}{m_2 - m_1^2}, \quad m_1 = \left. \left( \frac{d}{dt} \phi(t) \right) \right|_{t=0}, \quad m_2 = \left. \left( \frac{d^2}{dt^2} \phi(t) \right) \right|_{t=0},
\]

the first two moments of the exact and near-exact distribution of \( W \) will coincide (see [3]). The values of \( m_1 \) and \( m_2 \) can be computed numerically. The obtained near-exact distribution of \( W \) is a Generalized Near – Integer Gamma (GNIG) distribution (see [2] and [3]), \( GNIG(r_1, \ldots, r_{p-3}, r, \lambda_1, \ldots, \lambda_{p-3}, \mu; p-2) \), where \( \mu \) and \( r \) are given by (9), \( r_j = \lfloor (p-j-1)/2 \rfloor + \epsilon_j \) with \( \epsilon_j \) given by (8) and \( \lambda_j = \lfloor (n-2-j)/2 \rfloor, \ j=1, \ldots, p-3 \). This is the distribution of the sum of \( p-2 \) independent Gamma distributed random variables \( X_1, \ldots, X_{p-2} \), where \( X_j \) has integer shape parameter \( r_j \) and rate parameter \( \lambda_j > 0 \), with \( \lambda_j \neq \lambda_{j'} \), for all \( j, j' \in \{1, \ldots, p-3\} \) and \( X_{p-2} \) has noninteger shape parameter \( r \) and rate parameter
The probability density function and the cumulative distribution function of a GNIG distribution can be found in [3]. If for a given \( \alpha \), \( W^{\star(1-\alpha)} \) is the \( (1-\alpha) \) quantile of the obtained GNIG distribution, then since \( W = -\log L_k \), \( L_k^{\star(\alpha)} = e^{-W^{\star(1-\alpha)}} \) is the \( \alpha \) near-exact quantile of \( L_k \).

**NUMERICAL EXAMPLES AND SOME PERCENTAGE POINTS**

Let us consider the case \( n=10, \ p=5, \) and \( k=2 \). We compute the density of \( L_k \), using three different methods:

1. the expression of \( f_{l_k} \), as a G-function distribution given by equation (5), and using the mathematical software MAPLE. Its graph is denoted by \( f_{l_k}^{(1)} \).

2. We simulate 1 000 000 values of \( L_k \) as the product \( \prod_{j=1}^{4} X_j \), with \( X_j \sim \text{Beta}((5+j)/2,(4-j)/2), \ j=1,2, \ X_j \sim \text{Beta}((4+j)/2,(5-j)/2), \ j=3,4 \) as given by (3), and derive the density denoted by \( f_{l_k}^{(2)} \).

3. We simulate 1 000 000 values of the near-exact distribution of \( W \), i.e. the \( \text{GNIG}(2,1.0042;3.5,3.7824;3) \) distribution, as the distribution of the sum of three independent Gamma random variables, with shape parameters 2, 1 and 1.0042 and corresponding rate parameters 3.5, 3 and 3.7824. The values 1.0042 and 3.7824 for \( r \) and \( \mu \) are obtained numerically from (9). Using the formula \( W_k = e^{-W} \) we compute 1 000 000 “near-exact” realizations of \( L_k \) and derive a near-exact density denoted by \( f_{l_k}^{(3)} \).

Figure 1 show these three densities, which can be seen as almost identical.

![Figure 1. Density of \( L_k \), with \( n=10, \ p=5, \) and \( k=2 \), by three different approaches.](image-url)
The 0.05 and 0.01 significance points of the null distribution of $L_k$ were computed for various values of $n$, $p$ and $k$. The results are given in Table 1.

**Table 1. 1% and 5% points of $L_k$**

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<th>$p$</th>
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**CONCLUSIONS**

The computation of quantiles of a G-function distribution is still very slow when using precise numerical integration on the distribution tails. It is shown that by Monte Carlo simulation, quick and at the same time precise calculations can be achieved, especially when the number of variables $p$ is not very large. A near-exact distribution for the null distribution of $L_k$ is derived, which can be used for quick and accurate calculation of quantiles for all values of $p$.

**REFERENCES**


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