Soil structure interaction problem

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Abstract: *A model of infinite boundary soil-structure interaction problem is presented in the chapter. The structure is described by finite elements, the soil is described by partial differential equation of a hyperbolic type, and the contact between the soil and the structure is described by matrix integral equation. The structure and the soil damping are examined and a theorem for obtaining a structural finite element damping description is demonstrated. Presented approach is applied for modeling and simulation of structures and devices under dynamic and seismic loadings.*

Key words: *FEM; Structural Damping; Soil Structure Interaction Problem; Somiliana Numerical Solution for the Contact Zone, Active Bridge Control System.*

INTRODUCTION

Mathematical finite element description of the structure can be presented with matrix second order differential equation [1,16,17,18]:

$$\begin{bmatrix} \mathbf{M}_{yy} & \mathbf{M}_{yx} \\ \mathbf{M}_{xy} & \mathbf{M}_{xx} \end{bmatrix} . \begin{Bmatrix} \ddot{\mathbf{Y}} \\ \ddot{\mathbf{X}} \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_{yy} & \mathbf{C}_{yx} \\ \mathbf{C}_{xy} & \mathbf{C}_{xx} \end{bmatrix} . \begin{Bmatrix} \dot{\mathbf{Y}} \\ \dot{\mathbf{X}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{yy} & \mathbf{K}_{yx} \\ \mathbf{K}_{xy} & \mathbf{K}_{xx} \end{bmatrix} . \begin{Bmatrix} \mathbf{Y} \\ \mathbf{X} \end{Bmatrix} = \begin{Bmatrix} \mathbf{O} \\ \mathbf{R} \end{Bmatrix} , \qquad (1)$$

where $\mathbf{Y}$ represents matrix block of the structure unknown displacements, $\mathbf{X}$ represents matrix block of the known input structure displacements and $\mathbf{R}$ represents matrix block of the unknown dynamical support reactions. The blocking procedure in the equation (1) is conducted according the blocking of the displacement vector. The unknown displacements $\mathbf{Y}$ matrix block can be obtained from the first block equations of the expression (1):

$$\left[\mathbf{M}_{yy} \right] . \left\{ \ddot{\mathbf{Y}} \right\} + \left[\mathbf{C}_{yy} \right] . \left\{ \dot{\mathbf{Y}} \right\} + \left[\mathbf{K}_{yy} \right] . \left\{ \mathbf{Y} \right\} = \left[\mathbf{M}_{yx} \right] . \left\{ \ddot{\mathbf{X}} \right\} + \left[\mathbf{C}_{yx} \right] . \left\{ \dot{\mathbf{X}} \right\} + \left[\mathbf{K}_{yx} \right] . \left\{ \mathbf{X} \right\} . \qquad (2)$$

The Laplace image of the unknown block vector can be obtained by the formula:

$$\left\{ \mathbf{Y}(s) \right\} = \left[\mathbf{M}_{yy} . s^2 + \mathbf{C}_{yy} . s + \mathbf{K}_{yy} \right]^{-1} . \left[-\mathbf{M}_{yx} . s^2 - \mathbf{C}_{yx} . s - \mathbf{K}_{yx} \right] . \left\{ \mathbf{X}(\mathbf{s}) \right\} . \qquad (3)$$

ESSENTIAL OF STRUCTURAL DAMPING IN THE FINITE ELEMENT METHOD

Theorem: In the mechanical system frequency, equation $f(\omega) = 0$ possess only single roots $0 < \omega_1 < \omega_2 < ... < \omega_k < ... < \omega_n$, when the matrix

$$\left[\mathbf{W}(\omega_1, \omega_2, ..., \omega_k, ..., \omega_n) \right] =$$

$$= \begin{bmatrix} \omega_1 & \omega_1^3 & \omega_1^5 & ... & \omega_1^{2.n-1} \\ \omega_2 & \omega_2^3 & \omega_2^5 & ... & \omega_2^{2.n-1} \\ \omega_3 & \omega_3^3 & \omega_3^5 & ... & \omega_3^{2.n-1} \\ ... & ... & ... & ... & ... \\ \omega_n & \omega_n^3 & \omega_n^5 & ... & \omega_n^{2.n-1} \end{bmatrix} = \omega_1 . \omega_2 ... \omega_n . \begin{bmatrix} 1 & \omega_1^2 & \omega_1^4 & ... & \omega_1^{2.n-2} \\ 1 & \omega_2^2 & \omega_2^4 & ... & \omega_2^{2.n-2} \\ 1 & \omega_3^2 & \omega_3^4 & ... & \omega_3^{2.n-2} \\ ... & ... & ... & ... & ... \\ 1 & \omega_n^2 & \omega_n^4 & ... & \omega_n^{2.n-2} \end{bmatrix} \qquad (4)$$

is a positive determined, namely $\det \left[\mathbf{W}(\omega) \right] > 0$

Proof: Let us introduce the following linear algebraic matrix equation:

$$\left\{ \xi \right\} = \left[\mathbf{W} \right] . \left\{ \alpha \right\} . \qquad (5)$$

This equation links the model diagonal damping matrix $\left\{ \xi \right\}$, with the unknown general coordinates $\left\{ \alpha \right\}$, in the formula (5) (see [2,3]):

$$\left[\mathbf{C} \right] = \left[\mathbf{M} \right] . \sum_i a_i . \left(\left[\mathbf{M} \right]^{-1} . \left[\mathbf{K} \right] \right)^i . \tag{6}$$

The equation $\{\xi\} = [\mathbf{W}].\{\alpha\}$ is used for obtaining the global damping matrix $[\mathbf{C}]$ of the mechanical system finite element description. It always possesses nonzero solution:

$$\{\alpha\} = [\mathbf{W}]^{-1}.\{\xi\} . \tag{7}$$

The matrices in the finite element method description are symmetric and positively determined. The variables ω_i are real – see [4]. They are absolutely positive (natural frequency of the mechanical system under investigation). By the substitution $\omega_i^2 = x_i$, the matrix (4) is transformed into Wandermond mode. The determinant of this matrix:

$$\det\left[\mathbf{W}\left(\omega_1, \omega_2, ..., \omega_n \right) \right] = \begin{bmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ 1 & x_3 & x_3^2 & ... & x_3^{n-1} \\ ... & ... & ... & ... & ... \\ 1 & x_n & x_n^2 & ... & x_n^{n-1} \end{bmatrix} = \prod_{1 \le k < i \le n} \left(x_i - x_i \right) \tag{8}$$

is positive, because $x_i > x_k$; $\omega_k > 0$; $\omega_i > 0$; $x_k = \omega_k^2 > 0$; $x_i = \omega_i^2 > 0$ and $x_i \ne x_k$.

Consequently, the operator $[\mathbf{W}]^{-1}$ exists and the algebraic system (5) always possesses nonzero solution (7). The theorem is proved. The material damping and nonlinear deformations in the soil can be taken into account via the procedures, described in [13,14]. An original algorithm for obtaining material damping is presented in [11].

DYNAMIC CONDENSATION TECHNIQUE AND BLOCK INVERSING OF THE DYNAMICAL STIFFNESS MATRIX IN SOIL STRUCTURE INTERACTION PROBLEM

The global matrices in the discrete methods consist of a lot of elements. It is well known that the integrating of a large scale matrix differential system represents a difficult mathematical problem. Condensation methods are used for successful solving of this problem [5,6,7]. The condensation technique effectiveness depends on the matrices structure, of especial importance is width of the matrices band. Group of parameters (the initial unknown problem variables) are divided in to two sub-groups - master and slave by the dynamical condensation approach. The slave sub - group is reduced by corresponding procedures and the new condensed description of the system under investigation contains only master sub - group of parameters [5,6,7,8,15,16]. Generally there are two types of reducing procedures. The first type is characterized by the exact reducing of the slave sub – group [5,6,7,8,15,16]. The second type procedures are characterized by the modification of the initial system [9]. The aim of this modification is to guarantee equivalent dynamical behaviour of the initial and the modify systems. This system is obtained by convenient truncation of the frequency spectrum and corresponding transformation of the initial dynamical stiffness matrices. The truncated system has the same basic spectral properties as the original system. There are a lot of strategies for obtaining the master sub - group solution after applying condensation procedure. An algorithm for block inversing of the dynamical stiffness matrix $[\mathbf{S}(x)]$ is presented in this paragraph. The unknown inverse matrix is denoted by $[\mathbf{X}(x)]$. This inverse matrix is obtained from the definition equation:

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} . \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{21} & \mathbf{X}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} . \tag{9}$$

The unknown blocks $\mathbf{X}_{11}$ and $\mathbf{X}_{21}$ can be obtained from the following matrix system:

$$\begin{aligned} &\mathbf{S}_{11}.\mathbf{X}_{11}+\mathbf{S}_{12}.\mathbf{X}_{21}=\mathbf{I}\ ,\\ &\mathbf{S}_{21}.\mathbf{X}_{11}+\mathbf{S}_{22}.\mathbf{X}_{21}=\mathbf{O}\ , \end{aligned} \tag{10}$$

and the unknown blocks $\mathbf{X}_{12}$ and $\mathbf{X}_{22}$ can be obtained from the following matrix system:

$$\begin{aligned} &\mathbf{S}_{11}.\mathbf{X}_{12}+\mathbf{S}_{12}.\mathbf{X}_{22}=\mathbf{O}\ ,\\ &\mathbf{S}_{21}.\mathbf{X}_{12}+\mathbf{S}_{22}.\mathbf{X}_{22}=\mathbf{I}\ . \end{aligned} \tag{11}$$

By solving the above mentioned two systems the following blocks are obtained the unknown blocks in the initial description:

$$\begin{aligned} &\mathbf{X}_{11}=\left(\mathbf{S}_{11}-\mathbf{S}_{12}.\mathbf{S}_{22}^{-1}.\mathbf{S}_{21}\right)^{-1}\ ,\\ &\mathbf{X}_{12}=\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}.\left(\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}-\mathbf{S}_{22}\right)^{-1}\ ,\\ &\mathbf{X}_{21}=\left(\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}-\mathbf{S}_{22}\right)^{-1}.\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}\ ,\\ &\mathbf{X}_{22}=\left(\mathbf{S}_{22}-\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}\right)^{-1}\ . \end{aligned} \tag{12}$$

Inverse matrix can be written by using the above obtained blocks:

$$\left[\mathbf{S}\right]=\begin{bmatrix} \left(\mathbf{S}_{11}-\mathbf{S}_{12}.\mathbf{S}_{22}^{-1}.\mathbf{S}_{21}\right)^{-1} & \mathbf{S}_{11}^{-1}.\mathbf{S}_{12}.\left(\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}-\mathbf{S}_{22}\right)^{-1} \\ \left(\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}-\mathbf{S}_{22}\right)^{-1}.\mathbf{S}_{21}.\mathbf{S}_{11}^{-1} & \left(\mathbf{S}_{22}-\mathbf{S}_{21}.\mathbf{S}_{11}^{-1}.\mathbf{S}_{12}\right)^{-1} \end{bmatrix}. \tag{13}$$

With appropriate selection of the blocks in mathematical model (9) ("structure" and "soil" parameters), the equations (12) could be used for accurate solution of the soil structure interaction problem. The inverse matrix (13) formally coincides with the condensed description in dynamical condensed method [5].

COMPLEX STIFFNESS MATRIX OF THE INFINITE RIGID FOUNDATION, BASED ON THE ELASTIC HALF-SPACE

According to [10] the relationship between forces and displacements in the contact boundary zone of the rigid circle foundation (Fig.1.a), based on the surface of elastic half–space (Fig.1.b), can be assumed by the expression:

$$\begin{Bmatrix} P_x \\ M_\phi \\ P_z \\ M_\theta \end{Bmatrix}=\begin{bmatrix} K_x(a_0) & 0 & 0 & 0 \\ 0 & K_\phi(a_0) & 0 & 0 \\ 0 & 0 & K_z(a_0) & 0 \\ 0 & 0 & 0 & K_\theta(a_0) \end{bmatrix}.\begin{Bmatrix} u_x \\ u_\phi \\ u_z \\ u_\theta \end{Bmatrix}=\left[\mathbf{K}(a_0)\right].\left\{\mathbf{u}\right\}\ . \tag{14}$$

The elements of the complex stiffness matrix $\left[\mathbf{K}(a_0)\right]$ in equation (14) are obtained by the formulas:

$$\begin{aligned} &K_x=K_x^0.\left[k_x(a_0)+i.a_0.c_x(a_0)\right]\ ,\\ &K_\phi=K_\phi^0.\left[k_\phi(a_0)+i.a_0.c_\phi(a_0)\right]\ ,\\ &K_z=K_z^0.\left[k_z(a_0)+i.a_0.c_z(a_0)\right]\ ,\\ &K_\theta=K_\theta^0.\left[k_\theta(a_0)+i.a_0.c_\theta(a_0)\right]\ . \end{aligned} \tag{15}$$

The static stiffness parameters and non-dimensional frequency parameter are:

$$K_x^0 = \frac{8.G.r}{2-\nu}\ , \quad K_\phi^0 = \frac{8.G.r}{3.(1-\nu)}\ , \quad K_z^0 = \frac{4.G.r}{1-\nu}\ ,$$

$$K_\theta^0 = \frac{16.G.r^3}{2-\nu}\ , \quad a_0 = \frac{\omega.r}{c.s}\ . \tag{16}$$

In the above k_ϕ presented formulas c_s is the speed of propagation of the S wave in half space, ω is circular frequency of the excitation and r is the disc radius. The symbols $k_x,\ k_\varphi,\ k_z,\ k_\theta;\ c_x,\ c_\varphi,\ c_z$ and c_θ represent un - dimensioned functions of Poison coefficient and a_0 is un - dimensioned frequency parameter.

BOUNDARY INTEGRAL EQUATION DESCRIPTION OF THE SOIL

Equation of motion for continuous medium can be written in displacement mode as follows:

$$\left(V_1^2 - V_2^2\right).u_{i,ij} + V_2^2.u_{j,ii} + \frac{1}{\rho}.b_i = \ddot{u}_j\ , \tag{17}$$

where comma represents space derivation, dots represent time derivation, $V_1 = \sqrt{\frac{(\lambda + 2.\mu)}{\rho}}$ represents longitudinal wave velocity $V_2 = \sqrt{\frac{\mu}{\rho}}$ represents transversal wave velocity, $\lambda = \frac{E.r}{(1+\nu).(1-2.\nu)}$ and $\mu = G = \frac{E}{2.(1+\nu)}$ are Lame constants and ρ represents volume density.

By Laplace transformation equation (17) becomes:

$$\left(V_1^2 - V_2^2\right).U_{i,ij} + V_2^2.U_{j,ii} + \frac{Q_j}{\rho} = s^2.U_j\ , \tag{18}$$

where $U_i(x,s) = L\left[u_i(x,t)\right]$ and $B_i(x,s) = L\left[b_i(x,t)\right]$ are Laplace transformations of the is signified the following expression: $Q_j = B_j + \rho(s.u_j^0 + v_j^0)$ where u_j^0 and v_j^0 are initial problem conditions.

Fig.1.b shows the discrete model of the soil under the structure. The dynamical behaviour of the corresponding continuous model can be described in the complex domain by the following matrix boundary integral equation (Somiliana identity):

$$[\zeta].\{\mathbf{U}_f\} + \int_\Gamma \left[\mathbf{P}^*\right].\{\mathbf{U}\}.d\Gamma = \int_\Gamma \left[\mathbf{U}^*\right].\{\mathbf{P}\}.d\Gamma + \int_\Omega \left[\mathbf{U}^*\right].\{\mathbf{P}\}.d\Omega\ , \tag{19}$$

where $\{\mathbf{U}_f\}$ represents known excitation vector in the earthquake fireplace point (Fig.1.b), and $[\zeta]$ represents boundary shape matrix. The last term of this equation is vanished in the case of neglecting the inertial effects and equation (19) becomes in kinematic mode:

$$[\zeta].\{\mathbf{U}_f\} + \int_\Gamma \left[\mathbf{P}^*\right].\{\mathbf{U}\}.d\Gamma = \int_\Gamma \left[\mathbf{U}^*\right].\{\mathbf{P}\}.d\Gamma\ . \tag{20}$$

For the half space discrete (Fig.1.b, Fig.1.d) and continuous models (Fig.1.c, Fig.1.e) the fundamental stress tensor $\left[\mathbf{P}^*\right]$ is zero in the free surface [15] and the first integral of equation (20) is vanished too. The force vector in the second integral in expression (20) can be represented through the displacements by the formula (14).

Because of symmetry of the problem under investigation the unknown vector $\{\mathbf{U}\}$ is constant and goes out under the integral symbol:

$$[\zeta].\{\mathbf{U}_f\} = \int_\Gamma [\mathbf{U}^*].d\Gamma.\{[\mathbf{K}].\{\mathbf{U}\}\}.d\Gamma \ . \tag{21}$$

The fundamental displacements for elastic half-space can be constructed by the formula (12):

$$[\mathbf{U}^*] = [\mathbf{U}^k] + [\mathbf{U}^{a.d.d}] \ . \tag{22}$$

where $[\mathbf{U}^k]$ represents Kelvin's fundamental solution for displacements in the space and $[\mathbf{U}^{a.d.d}]$ represents additional fundamental solution for displacements in elastic half space model according to [12].

NUMERICAL SOLUTION FOR THE CONTACT ZONE

The analytical fundamental solution (22) is very complex and the analytical integration in the formula (21) is quite difficult. A satisfactory numerical solution can be obtained by the finite element model of the whole system (Fig.1.e). The template soil stiffness matrix for axis symmetric problem ($M_\theta = 0$) is shown in Table 1.

									Table 1
	I					I			
I		$u_x\,[m]$	$u_z\,[m]$	$\phi_x\,[rad]$	I		$u_x\,[m]$	$u_z\,[m]$	$\phi_x\,[rad]$
	$R_x\,[kN]$ R_x	*	0	0		$R_x\,[kN]$	*	0	0
1	$R_z\,[kN]$	*	*	0	1	$R_z\,[kN]$	*	*	0
	$M\,[kN.m]$	0	0	0		$M\,[kN.m]$	0	0	0

Another satisfactory numerical solution can be obtained by application the unit forces in the earthquake fireplace point in the middle of the model (Fig.1.b). Pertinent results for the axis symmetric problem ($M_\theta = 0$) are collected in Table 4 and Table 5. This result can be replaced in the formula (21) in capacity of numerical fundamental solution $\{\mathbf{U}^*\}$. After integrating the algebraic system according unknown images of displacements $\{\mathbf{U}\}$ can be assembled with the master structural stiffness matrices. By a similar procedure the numerical finite element soil solution can be assembled with the master structure stiffness matrix for obtaining the global stiffness matrix of the model. Corresponding global mass matrices can be assembled by the same procedure and the global damping matrix can be obtained by the formula (6) from the proofed theorem.

ILLUSTRATIVE EXAMPLES

The dynamic behavior of a discrete RTV (radio-television) tower model is study (Fig.1.c). This structure is developed for the aims of Bulgarian town Plovdiv. A vertical cross-section of the tower model is shown in Fig.1.d. A master stiffness matrix for the system under investigation is obtained via the described in the chapter dynamic condensation procedure and is presented in Table 2. The corresponding the indications are: Level 1 means the foundation of the tower; Level 2 means the end of the reinforced concrete part of the tower; Level 3 means the still top of the tower. Appropriate natural frequencies for the structure under investigation are given in Table 3.

Table 2

L		11-level			2 -level			3 -level		
	Reaction Displacement	$U_x\ [m]$	$U_z\ [m]$	$\phi_x\ [rad]$	$U_x\ [m]$	$U_z\ [m]$	$\phi_x\ [rad]$	$U_x\ [m]$	$U_z\ [m]$	$\phi_x\ [rad]$
1	$R_x\ [kN]$	6.5*10^4	3*10^3	-150	-6.5*10^{-4}	0	-5	106	0	0
	$R_z\ [kN]$	3*10^3	2.3*10^6	6	0	-2.3*10^6	1	0	-4*10^4	0
	$M_x\ [kN.m]$	-150	6	1.6*10^6	-150	-2	6.5*10^6	0	0	-4.2*10^3
2	$R_x\ [kN]$	-6.5*10^4	0	-150	6.5*10^4	0	5	-260	0	0
	$R_z\ [kN]$	0	-2.3*10^6	-2	0	2.3*10^6	-1	0	-4.5*10^4	0
	$M_x\ [kN.m]$	-5	1	6.5*10^4	5	-1	6.6*10^6	0	0	635
3	$R_x\ [kN]$	106	0	0	-260	0	0	150	0	0
	$R_z\ [kN]$	0	-4*10^4	0	0	-4.5*10^4	0	0	8.5*10^4	0
	$M_x\ [kN.m]$	0	0	-4.2*10^3	0	0	635	0	0	1.4*10^4

Table 3

N	$f\ [Hz]$	$\omega\ [1/s]$	$T\ [s]$
1	2.12	13.326	0.470
2	2.59	16.331	0.381
3	7.92	49.763	0.126
4	8.08	50.771	0.123
5	11.73	73.705	0.085
6	14.16	88.971	0.071

Table4

	L			
I		$u_x[m]$s	$u_z[m]$	$\phi_x\ [rad]$
	$R_x\ [kN]$	3.1*10^6	0	0
1	$R_z\ [kN]$	0	4.4*10^6	0
	$M\ [kN.m]$	0	0	152*10^6

Table 5

	L			
I		$u_x[m]$	$u_z[m]$	$\phi_x\ [rad]$
	$R_x\ [kN]$	-2.2*10^{-9}	0	0
1	$R_z\ [kN]$	4.4*10^{-8}	1.7*10^{-8}	0
	$M\ [kN.m]$	0	0	0

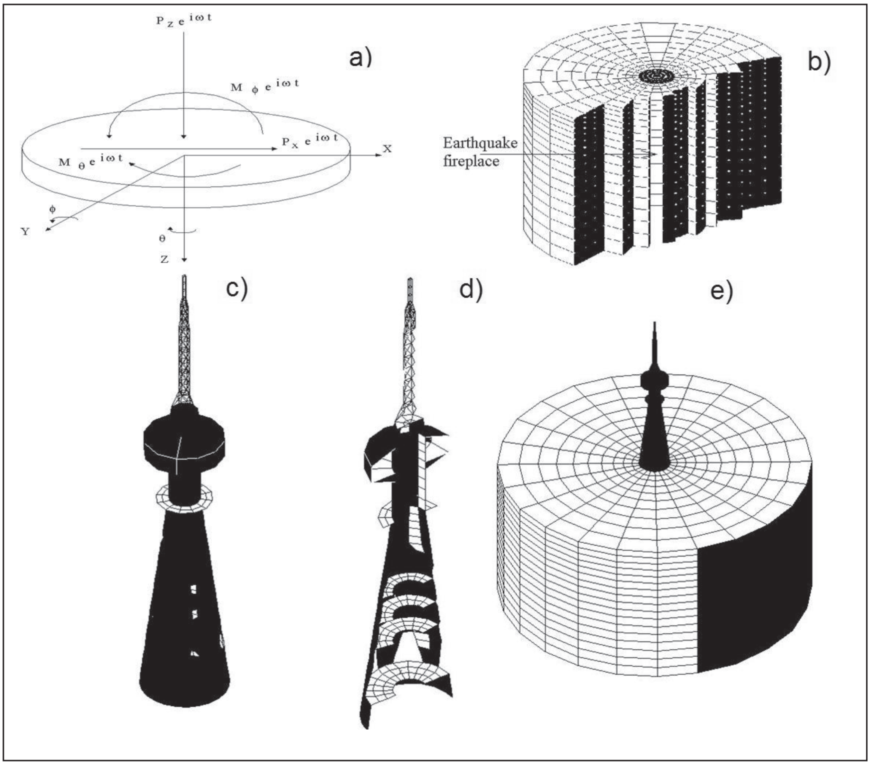

Fig.1. Soil structure interaction model:

a) Rigid circular foundation on the surface of an elastic half-space;
b) Discrete model of soil under the structure;
c) Finite element model for the RTV tower;
d) Vertical cross-section of the RTV tower;
e) Soil structure interaction model.

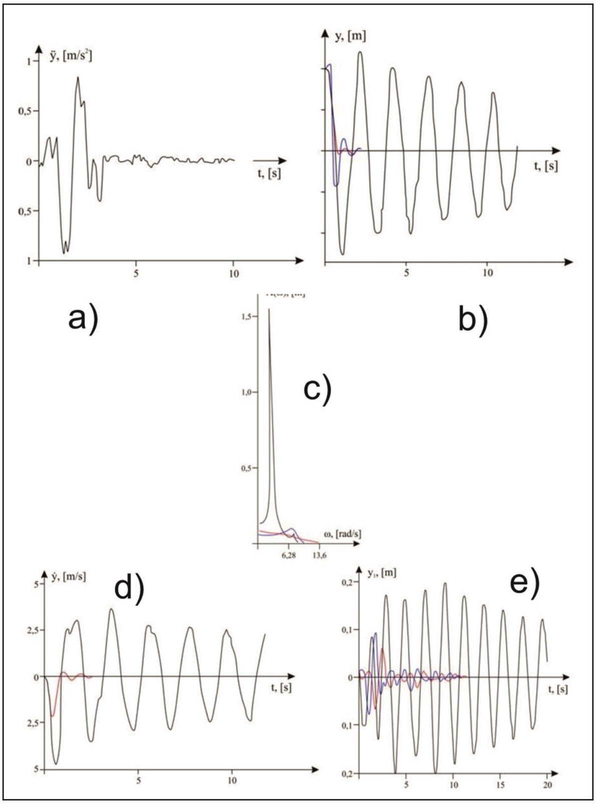

Fig.2. Frequency and time domain characteristics of the RTV tower:

a) Input seismic signal – accelerogram from Bucharest, 1977 (NS-component);
b) Free motion - tower antenna displacement:
without active control *(black)*, with optimal control *(red),* with modal control *(blue)*;
c) Free motion - tower antenna velocity:
without active control *(black),* with optimal control *(red)*;
d) Relative displacement of the tower antenna to Earth acceleration – frequency characteristics: without active control *(black)*, with optimal control *(red),*
with modal control *(blue)*;
e) Forced motion; tower antenna displacement:
without active control *(black),* with optimal control *(red),* with modal control *(blue)*.

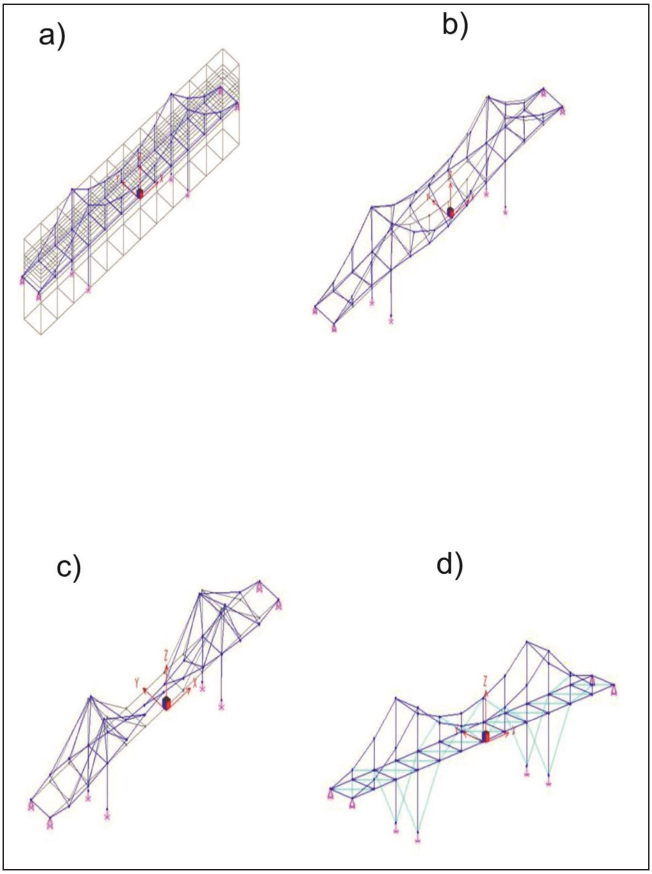

Fig.3. Idea solution of Active Bridge Control System:

a) Un-deformed shape of large scale suspended bridge;
b) Deformed torsion mode of vibration suspended bridge;
c) Deformed torsion mode of vibration cable state bridge;
d) Dynamic equilibrium of active control bridge system.

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