Bivariate Pòlya-Aeppli risk model*

Krasimira Kostadinova

Abstract: *In this paper we consider a risk model in which the claim counting process is the bivariate Pòlya-Aeppli process, defined by Minkova and Balakrishnan in [9]. We call it a bivariate Pòlya-Aeppli risk model. We also consider two types of ruin probability for this risk model and find the Laplace transform for the ruin probability. We investigate in detail the particular case of exponentially distributed claims.*

Key words: *Ruin probability, Pòlya-Aeppli process, bivariate geometric distribution.*

1. INTRODUCTION

In many cases, the claims to the insurance company can arrive simultaneously. For example, an accident can cause two claims simultaneously: for motor insurance and for personal injury. Thus, for modelling we need some bivariate risk model.

The surplus process of an insurance company is described by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Z_i, \ \left(\sum_{1}^{0} = 0\right), \ t \geq 0, \tag{1}$$

where u is the initial capital, c represents the premium income per unit time and $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ ($N(t) \sim Po(\lambda t)$). Z_i, $i = 1, 2, \dots$ is a sequence of independent, identically distributed random variables (iid r.v.'s), independent of the counting process $N(t)$.

Mostly, a compound Poisson process is used for the counting process in the process $U(t)$. A compound Poisson process can be represented by the sum

$$N(t) = X_1 + X_2 + \dots + X_{N_1(t)}, \tag{2}$$

where $N_1(t) \sim Po(\lambda t)$ and $X_1, X_2, \dots$ are iid r.v.'s, independent of $N_1(t)$, see [4], [6], [7] and [8]. The probability generating function (PGF) of the process $N(t)$ in (2) is given by

$$\psi_{N(t)}(s) = e^{-\lambda t(1-\psi_1(s))}, \tag{3}$$

where $\psi_1(s)$ is the PGF of the compounding random variable X. In the case of geometric compounding distribution, i.e. $X_i \sim Ge_1(1-\rho)$, $i = 1, 2, \dots$, $N(t)$ has a Pòlya-Aeppli distribution with parameters λt and ρ ($N(t) \sim PA(\lambda t, \rho)$). The defined process is called a Pòlya-Aeppli process. For the Pòlya-Aeppli process we use the notation $N(t) \sim PAP(\lambda, \rho)$, see [2]. The probability mass function (PMF) is given in the next theorem (see [3] and [8]).

Theorem 1.1. The probability mass function of $PAP(\lambda, \rho)$ is given by

$$P(N(t)=i) = \begin{cases} e^{-\lambda t}, i = 0, \\ e^{-\lambda t} \sum_{m=1}^{i} \binom{i-1}{m-1} \dfrac{(\lambda(1-\rho))^m}{m!} \rho^{i-m}, i = 1,2,\dots \end{cases} \tag{4}$$

In this paper we suppose that the compounding r.v. $X = (X_1, X_2)$ has a bivariate geometric distribution, i.e. we introduce a process with Type II Bivariate Pòlya-Aeppli distribution. The Type II Bivariate Pòlya-Aeppli distribution was defined by Minkova and Balakrishnan in [9]. The PGF of the bivariate geometric distribution ($BivGe(\alpha, \beta)$), see [5], is given by

$$\psi_1(s_1, s_2) = \frac{\gamma}{1 - \alpha s_1 - \beta s_2}, \tag{5}$$

where α and β are nonnegative parameters and $\gamma = 1 - \alpha - \beta$. Then we define a bivariate process $(N_1(t), N_2(t))$ with joint PGF given by

* This work is supported by grant RD-08-265/2015 of Shumen University, Bulgaria

$$\psi(s_1, s_2) = e^{-\lambda t(1-\psi_1(s_1,\ s_2))}. \qquad (6)$$

We can give the following definition of Type II Bivariate Pòlya-Aeppli process and its PMF.

Definition 1.1. The bivariate process *($N_1(t)$, $N_2(t)$)*, corresponding to (6), is called Type II bivariate Pòlya-Aeppli process (*BPAP(λ, α, β)*), with parameters λ, α and β.

Theorem 1.2. The probability mass function of *BPAP(λ, α, β)* process is given by

$$\begin{aligned} f(0,\ 0) &= e^{-\lambda t(1-\gamma)}, \\ f(i,\ j) &= \binom{i+j}{j}\alpha^i\beta^j\sum_{m=1}^{\infty}\binom{m+i+j-1}{i+j}\frac{(\lambda t\gamma)^m}{m!}e^{-\lambda t},\ i,\ j = 0, 1,...,\ (i,\ j)\neq(0,\ 0). \end{aligned} \qquad (7)$$

2. BIVARIATE RISK MODEL

Let us consider the following bivariate surplus process for two lines of business

$$\begin{aligned} U_1(t) &= u_1 + c_1 t - \sum_{j=1}^{N_1(t)} Z_j^1 \\ U_2(t) &= u_2 + c_2 t - \sum_{j=1}^{N_2(t)} Z_j^2 \end{aligned} \qquad (8)$$

Here u_1 and u_2 are the initial capitals; c_1 and c_2 represent the premium incomes per unit time and $Z^1,\ Z_1^1,\ Z_2^1,\ \dots$, and $Z^2,\ Z_1^2,\ Z_2^2,\ \dots$ are two independent sequences of iid r.v.'s, independent of the counting processes $N_1(t)$ and $N_2(t)$, representing the corresponding claim sizes. Let $\mu_1 = E(Z^1)$ and $\mu_2 = E(Z^2)$ be the means of the claims. Denote by $S_1(t) = \sum_{j=1}^{N_1(t)} Z_j^1$ and $S_2(t) = \sum_{j=1}^{N_2(t)} Z_j^2$ the corresponding accumulated claim processes.

We consider (8), where $N_1(t)$ and $N_2(t)$ are Pòlya-Aeppli counting processes and we call this model a bivariate Pòlya-Aeppli risk model. The case of $N_1(t) = N_2(t) = N(t)$ is analysed in [1].

In this note we are going to consider two possible times to ruin

$$\tau_{max} = inf\{\ t \mid max(U_1(t), U_2(t)) < 0\} \qquad (9)$$

and

$$\tau_{sum} = inf\{\ t \mid U_1(t) + U_2(t) < 0\}, \qquad (10)$$

and the corresponding ruin probabilities

$$\psi_{max}(u_1, u_2) = P(\tau_{max} < \infty) \qquad (11)$$

and

$$\psi_{sum}(u_1, u_2) = P(\tau_{sum} < \infty). \qquad (12)$$

For the event in (9) we have the following:

$$\{max(U_1(t), U_2(t)) < 0\} = \{U_1(t) < 0, U_2(t) < 0\} = \{S_1(t) > u_1 + c_1 t, S_2(t) > u_2 + c_2 t\}.$$

It follows that the ruin probability $\psi_{max}(u_1, u_2)$ is the joint survival function of $(S_1(t), S_2(t))$. In a similar way, for the event in (10) we obtain

$$\{U_1(t) + U_2(t) < 0\} = \{S_1(t) + S_2(t) > u_1 + u_2 + (c_1 + c_2)t\}, \qquad (13)$$

i.e., the ruin probability $\psi_{sum}(u_1, u_2)$ is the survival function of the sum $S_1(t) + S_2(t)$.

2.1. LAPLACE TRANSFORMS

Denote by $LT_{Z^1}(s_1)$ and $LT_{Z^2}(s_2)$ the Laplace transforms of the r.v.'s Z^1 and Z^2. Then, the Laplace transform of $(S_1(t), S_2(t))$ is given by

$$\begin{aligned} LT_{(S_1(t),\ S_2(t))}(s_1,\ s_2) &= E[e^{-s_1 S_1(t)-s_2 S_2(t)}] = \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} E[e^{-s_1(Z_1^1+\ldots+Z_i^1)\ -s_2(Z_1^2+\ldots+Z_j^2)}]P(N_1(t)=i,\ N_2(t)=j) = \\ &= \psi_{(N_1(t),N_2(t))}(LT_{Z^1}(s_1),\ LT_{Z^2}(s_2)) = e^{-\lambda t\left[1-\frac{\gamma}{1-\alpha LT_{Z^1}(s_1)-\beta LT_{Z^2}(s_2)}\right]}, \end{aligned} \tag{14}$$

where $\gamma = 1-\alpha-\beta$.

The Laplace transforms of the marginal compounding distributions are given by

$$LT_{S_1(t)}(s_1) = LT_{(S_1(t),S_2(t))}(s_1,0) = e^{-\lambda t\left[1-\frac{1-\rho_1}{1-\rho_1 LT_{Z^1}(s_1)}\right]},\ \rho_1 = \frac{\alpha}{1-\beta} \tag{15}$$

and

$$LT_{S_2(t)}(s_2) = LT_{(S_1(t),S_2(t))}(0,s_2) = e^{-\lambda t\left[1-\frac{1-\rho_2}{1-\rho_2 LT_{Z^2}(s_2)}\right]},\ \rho_2 = \frac{\beta}{1-\alpha}. \tag{16}$$

For the ruin probability ψ_{sum} we have

$$LT_{S_1(t)+S_2(t)}(s) = LT_{(S_1(t),\ S_2(t))}(s,\ s) = e^{-\lambda t\left[1-\frac{\gamma}{1-\alpha LT_{Z^1}(s)-\beta LT_{Z^2}(s)}\right]}. \tag{17}$$

In the next lemma we use a result related to Laplace transforms, given in [10].

Lemma 2.1. For the joint survival function $P(S_1(t) > x,\ S_2(t) > y)$ we have

$$\begin{aligned} &\int_0^\infty\int_0^\infty e^{-s_1x-s_2y} P(S_1(t)>x,\ S_2(t)>y)dxdy \\ &= \frac{1-LT_{S_1(t)}(s_1)-LT_{S_2(t)}(s_2)+LT_{(S_1(t),S_2(t))}(s_1,\ s_2)}{s_1s_2} \\ &= \frac{1}{s_1s_2}\left[1-e^{-\lambda t\left[1-\frac{1-\rho_1}{1-\rho_1 LT_{Z^1}(s_1)}\right]}-e^{-\lambda t\left[1-\frac{1-\rho_2}{1-\rho_2 LT_{Z^2}(s_2)}\right]}+e^{-\lambda t\left[1-\frac{\gamma}{1-\alpha LT_{Z^1}(s_1)-\beta LT_{Z^2}(s_2)}\right]}\right]. \end{aligned} \tag{18}$$

Lemma 2.2. For the survival function $P(S_1(t) + S_2(t) > x)$ we obtain

$$\int_0^\infty e^{-sx}P(S_1(t)+S_2(t)>x)dx = \frac{1}{s}[1-LT_{S_1(t)+S_2(t)}(s)] = \frac{1}{s}\left[1-e^{-\lambda t\left[1-\frac{\gamma}{1-\alpha LT_{Z^1}(s)-\beta LT_{Z^2}(s)}\right]}\right]. \tag{19}$$

2.2. EXPONENTIALLY DISTRIBUTED CLAIMS

Let us consider the case of exponentially distributed claim sizes, i.e. $F_{Z^1}(x) = 1-e^{-\frac{x}{\mu_1}}$, $x \geq 0$ and $G_{Z^2}(y) = 1-e^{-\frac{y}{\mu_2}}$, $y \geq 0$, and $\mu_1, \mu_2 > 0$. Denote by

$$e(n,x) = \sum_{k=0}^{n}\frac{x^k}{k!} = \frac{e^x\Gamma(n+1,x)}{\Gamma(n+1)}, \tag{20}$$

a truncated exponential sum function, where *Γ(n)* is a Gamma function and $\Gamma(a,\ x)=\int_x^{\infty} t^{a-1}e^{-t}dt$ is the incomplete Gamma function. For the ruin probability ψ_{max} we have

$$P(S_1(t)>x,\ S_2(t)>y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\overline{F}^{*i}(x)\overline{G}^{*j}(y)P(N_1(t)=i,\ N_2(t)=j), \tag{21}$$

where

$\overline{F}^{*i}(x)=e^{-\frac{x}{\mu_1}}e\left(i-1,\ \frac{x}{\mu_1}\right)$, *i = 1, 2, ...* is the tail distribution of $Z_1^1+...+Z_i^1$ and $\overline{G}^{*j}(y)=e^{-\frac{y}{\mu_2}}e\left(j-1,\ \frac{y}{\mu_2}\right)$, *j = 1, 2, …* is the tail distribution of $Z_1^2+...+Z_j^2$.

In this case we have

$$\begin{aligned}
&P(S_1(t)>x,\ S_2(t)>y)\\
&=e^{-\lambda t(1-\gamma)}+\sum_{i=1}^{\infty}e\left(i-1,\ \frac{x}{\mu_1}\right)\alpha^i\sum_{m=1}^{\infty}\binom{m+i-1}{i}\frac{(\lambda t\gamma)^m}{m!}e^{-\frac{x}{\mu_1}}e^{-\lambda t}\\
&+\sum_{j=1}^{\infty}e\left(j-1,\ \frac{y}{\mu_2}\right)\beta^j\sum_{m=1}^{\infty}\binom{m+j-1}{j}\frac{(\lambda t\gamma)^m}{m!}e^{-\frac{y}{\mu_2}}e^{-\lambda t}\\
&+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha^i\beta^j e\left(i-1,\ \frac{x}{\mu_1}\right)e\left(j-1,\ \frac{y}{\mu_2}\right)\binom{i+j}{j}\sum_{m=1}^{\infty}\binom{m+i+j-1}{i+j}\frac{(\lambda t\gamma)^m}{m!}e^{-\frac{x}{\mu_1}-\frac{y}{\mu_2}}e^{-\lambda t}.
\end{aligned} \tag{22}$$

Substituting $x = u_1 + c_1t$ and $y = u_2 + c_2t$ in (22), we obtain the ruin probability $\psi_{max}(u_1, u_2)$.

For the ruin probability $\psi_{sum}(u_1, u_2)$ in the case of $Z^1 = Z^2 = Z$, and hence $\mu_1 = \mu_2 = \mu$, we obtain the Laplace transform of the sum $S_1(t) + S_2(t)$

$$LT_{S_1(t)+S_2(t)}(s)=e^{-\lambda t\left[1-\frac{\gamma}{1-(\alpha+\beta)LT_Z(s)}\right]}. \tag{23}$$

This means that

$$S_1(t) + S_2(t) = Z_1 + \dots + Z_{N(t)}, \tag{24}$$

where *N(t)* has a modified Pòlya-Aeppli distribution with parameters $\lambda t > 0$ and $\gamma\in[0,1)$ with PMF

$$P(N(t)=i)=\begin{cases} e^{-(1-\gamma)\lambda t}, i=0,\\ e^{-(1-\gamma)\lambda t}(1-\gamma)^i\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{(\gamma\lambda t)^j}{j!}, i=1,2,... \end{cases} \tag{25}$$

The survival function of the sum $S_1(t) + S_2(t)$ is the survival function of the sum of claims, i.e.

$$P(S_1(t)+S_2(t)>x)=P(Z_1+...+Z_{N(t)}>x)=\sum_{i=0}^{\infty}\overline{F}^{*i}(x)P(N(t)=i), \tag{26}$$

where $\overline{F}^{*i}(x)=e^{-\frac{x}{\mu}}e\left(i-1,\ \frac{x}{\mu}\right)$, *i = 1, 2, ...* is the tail distribution of $Z_1 + \dots + Z_i$. Hence we have

$$\begin{aligned}
P(S_1(t)+S_2(t)>x)&=e^{-(1-\gamma)\lambda t}+\sum_{i=1}^{\infty}\overline{F}^{*i}(x)\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{(\gamma\lambda t)^j}{j!}(1-\gamma)^i e^{-(1-\gamma)\lambda t}\\
&=e^{-(1-\gamma)\lambda t}+\sum_{i=1}^{\infty}e\left(i-1,\ \frac{x}{\mu}\right)(1-\gamma)^i\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{(\gamma\lambda t)^j}{j!}e^{-\frac{x}{\mu}}e^{-(1-\gamma)\lambda t}
\end{aligned} \tag{27}$$

and for $x = u_1 + u_2 + (c_1 + c_2)t$, we obtained the ruin probabilities ψ_{sum}.

3. CONCLUSIONS

In this study we define a bivariate Pòlya-Aeppli risk model. We introduce two types of ruin probability for the defined risk model. We also obtain the Laplace transform of the ruin probability and investigate a special case of exponentially distributed claims.

REFERENCES

[1] Chan, W-S, H. ang, L. Zhang. Some results on ruin probabilities in a two-dimensional risk model. Insurance Mathematics&Economics 33, 345-358, 2003.

[2] Chukova, S., L.D. Minkova. Characterization of the Pòlya-Aeppli process. Stochastic Analysis and Applications 31, 590-599, 2013.

[3] Johnson, N., A. Kemp, S. Kotz. Univariate Discrete Distributions. Wiley Series in Probability and Mathematical Statistics, 3th edition, 2005.

[4] Klugman, S., H. Panjer, G. Willmot. Loss Models: From Data to Decisions. John Wiley&Sons,Inc., 1998.

[5] Kocherlakota, S., K. Kocherlakota. Bivariate Discrete Distributions. Marcel Dekker, New York, 1992.

[6] Kostadinova, K. On a Poisson Negative Binomial Process. Advanced Research in Mathematics and Computer Science, Doctoral Conference in Mathematics, Informatics and Education, September, 19-21, Sofia, 25-33, 2013.

[7] Kostadinova, K., L. Minkova. On the Poisson process of order *k*. Pliska Studia Mathematica Bulgarica, vol. 22, pp. 117-128, 2013.

[8] Minkova, L. The Pòlya-Aeppli process and ruin problems. J. Appl. Math. Stoch. Anal. 3, 221-234, 2004.

[9] Minkova, L., N. Balakrishnan. Type II Bivariate Pòlya-Aeppli distribution. Statistics&Probability Letters 88, 40-49, 2014.

[10] Omey, E., L. Minkova. Bivariate Geometric Distributions. Compt. Randue Bulg. Acad. Sci. 67(9), 1201-1210, 2014.

ABOUT THE AUTHOR

Assist.Prof. Krasimira Kostadinova, Faculty of Mathematics and Informatics, Shumen University, Phone: +359 54 830 495, E-mail: kostadinova@shu-bg.net.