A Generalization of Two Pure Birth Processes and their Applications to Insurance Risk Theory

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Abstract: Pure birth processes and their applications. *This paper introduces two different pure birth processes and some of their applications to risk models. The first one is the Poisson process of order k, which is a compound birth process and the second one is a mixed Pólya-Aepply process with shifted gamma mixing distribution, called Inflated –parameter Delaporte process (I-Delaporte). The counting process in the first risk model is the Poisson process of order k and the counting process in the second one is the I-Delaporte process. For these two risk models are given the join distribution of the time to ruin and the deficit at ruin.*

Key words: *compound Poisson process, mixed Pólya-Aepply process, ruin probability, nonruin probability, risk model*

1. ITRODUCTION

The foundation of the risk theory comes from the works of Filip Lundberg and Harald Cramér. In 1903 Filip Lundberg proposed the Poisson process as a simple process in solving the problem of first passage time. Later in 1930 Harald Cramér extended Lundberg's work, using it for modeling the ruin of an insurance company as a first passage time problem. That's why the basic model is called a Cramér-Lundberg model or classical risk model. The standard risk model of an insurance company, called risk process {*X(t)*, t≥0}, defined on the complete probability space (Ω,*F*,P), is given by the equation:

$$X(t) = c.t - \sum_{k=1}^{N(t)} Z_k \ , \quad \left(\sum_{1}^{0} = 0 \right) \tag{1}$$

where c is a positive real constant representing the *gross risk premium* rate and $N(t)$, t≥0 is a counting process. The process $N(t)$ is interpreted as the number of claims to the company and the risk process $X(t)$ represents the profit of the risky business during the interval (0,t]. The sequence $\{Z_k\}_{k=1}^{\infty}$ of non-negative independent and identically distributed random variables represents the amounts of successful claims to the insurance company. It is independent of the counting process $N(t)$, t≥0. The claim sizes $\{Z_k\}_{k=1}^{\infty}$ are distrubited as the random variable Z with distribution function $F,\ F(0)=0$ and mean value μ.

The accumulated sum of claims up to the time t is given by:

$$S(t) = \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0 \tag{2}$$

The process $S(t) = \left(S(t)\right)_{t\geq 0}$ is defined by the sum $S_n = Z_1 + Z_2 + \ldots + Z_n$, where n is a realization of the random variable $N(t)$. Also we can write:

$$S(t) = Z_1 + Z_2 + \ldots + Z_{N(t)} = S_{N(t)}, \quad t \geq 0 \ . \tag{3}$$

In the classical risk model, the process $N(t)$ is a stationary Poisson counting process, see [6]. The main interest in Insurance risk theory is mainly related to the counting process $N(t)$. This fact gave the idea to make a review of two counting processes and their applications in risk theory. The first risk model is called Poisson process of order k risk model, see [4] and the second one is I-Delaporte risk model, see [5].

2. DEFINITIONS

2.1. POISSON PROCESS OF ORDER K

Let us consider a stochastic process $N(t)$, *t>0* defined on a fixed probability space (Ω,*F*,P) and given by: $N(t) = X_1 + X_2 + \ldots + X_{N_1(t)}$, where X_i, $i = 1,2,\ldots$ are independent and identically distributed random variables, independent of $N_1(t)$. The counting process $N_1(t)$ is a Poisson process with intensity *k.λ>0 i.e.* $(N_1(t) \sim Po(k\lambda t))$. In this case $N(t)$ is a compound Poisson process, see [3]. The probability mass function (PMF) and the probability generating function (PGF) of $N_1(t)$ are given by: $P\left(N_1(t) = i\right) = \frac{(k\lambda t)^i . e^{-k\lambda t}}{i!}$, $i = 0,1,\ldots$ and $\psi_{N_1(t)}(s) = e^{-k\lambda t(1-s)}$.

Let the compounding random variable X be a discrete uniformly distributed over k>1 points with PMF: $P(X = i) = \frac{1}{k}$, $i = 1,2,\ldots,k$. The PGF of X is given by

$$\psi_X(s) = \frac{s}{k} . \frac{1 - s^k}{1 - s} . \tag{4}$$

Then for the PGF of the process $N(t)$ we get:

$$\psi_{N(t)}(s) = e^{-k\lambda t(1-\psi_X(s))}, \tag{5}$$

where $\psi_X(s)$ is the PGF of the compounding distribution**.**

Definition1: The stochastic process, defined by the PGF (5) and compounding distribution, defined by (4) is called a Poisson process of order k with parameter λ .We denote it $N(t) \sim Po_k(\lambda t)$.

Remark: If k=1, the discrete uniform distribution in (4) degenerates at point one, and the process $N(t)$ is a homogeneous Poisson process.

Moments: The mean and the variance of the Poisson process of order k are given by:

$$E(N(t)) = \frac{1+k}{2} k\lambda t \quad \text{and} \quad Var(N(t)) = \frac{(k+1).(2k+1)}{6} k\lambda t .$$

Conclusion: The Poisson process of order k is a compound Poisson process with discrete uniform compounding distribution, see [4]

Definition2: The stochastic process, defined by the following Kolmogorov's equations:

$$P_0'(t) = -k\lambda P_0(t)$$

$$P_m'(t) = -k\lambda P_m(t) + \lambda \sum_{j=1}^{m \wedge k} P_{m-j}(t), \quad m = 1,2,\ldots \tag{6}$$

with initial conditions $P_0(0) = 1$ and $P_m(0) = 0$, $m = 1,2,\ldots$ is called a Poisson process of order k.

Conclusion: The Poisson process of order k is a pure birth process, see [4]

2.2 I-DELAPORTE PROCESS

I-Delaporte process, also known as an Inflated-parameter Delaporte process is a mixed Pólya - Aepply process with shifted gamma mixing distribution. The parameter λ of this process has a shifted gamma distribution with density:

$g(\lambda) = \frac{\beta^r}{\Gamma(r)} (\lambda - \alpha)^{r-1} e^{-\beta(\lambda - \alpha)}$, where $\beta > 0$, $\lambda > a$, see [2]. After mixing, it turned out that the resulting process is a sum of two independent processes $N(t) = N_1(t) + N_2(t)$, see [4], where the process $N_1(t)$ has an Inflated-parameter negative binomial distribution with parameters $\frac{\beta}{\beta+t}, \rho$ and r, see [6]. Shortly we say I-Negative binomial and use the notation $N_1(t) \sim INB(\frac{\beta}{\beta+t}, \rho, r)$. The second process $N_2(t)$ has a Pólya–Aepply distribution with parameters λ and $\rho \in [0,1)$. For this process we use the notation $N_2(t) \sim PA(\lambda t, \rho)$. The PGF of the mixed process $N(t)$ is given by

$$\psi_{N(t)}(s) = \left[\frac{\pi}{1-(1-\pi)\psi_1(s)}\right]^r e^{-\alpha t(1-\psi_1(s))}, \tag{7}$$

where $\psi_1(s) = \mathrm{E}s^X = \frac{(1-\rho)s}{1-\rho s}$ is the probability generating function of the variable X, which has a geometric distribution with success probability $1-\rho$ i.e. $X \sim Ge_1(1-\rho)$, see [5]

Remark 1: In the case $\rho = 0$ the distribution of $N(t)$ simplifies to the Delaporte distribution, see [2]. In this case it is a composition of independent negative binomial and Poisson distributions.

Definition 1: The process $N(t)$, defined by the PGF (7) is called an Inflated-parameter Delaporte process (I-Delaporte process).

Moments: The mean and the variance of the I-Delaporte process are given by:

$EN(t) = \left(\alpha + \frac{r}{\beta}\right) \cdot \frac{t}{1-\rho}$ and $Var(N(t)) = \left[\alpha(1+p) + \frac{r((1+\rho)\beta + t}{\beta^2}\right] \frac{t}{(1-\rho)^2}$.

Definition2: The stochastic process, defined by the following Kolmogorov's equations:

$$P_0'(t) = -\left(\alpha + \frac{r}{\beta+t}\right) P_0(t),$$

$$P_m'(t) = -\left(\alpha + \frac{r}{\beta+t}\right) P_m(t) + (1-\rho)\sum_{k=1}^{m}\left[\alpha\rho^{k-1} + \frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^2 \left(1-(1-\rho)\frac{\beta}{\beta+t}\right)^{k-1}\right] P_{m-k}(t), \quad m = 1, 2, \ldots \tag{8}$$

with initial conditions $P_0(0) = 1$ and $P_m(0) = 0,\ m = 1, 2, \ldots$ is called I-Delaporte process.

Conclusion: I-Delaporte process is a pure birth process, see [5]

3. APPLICATION TO RISK THEORY

Giving the Application in Insurance risk theory of the processes above, we consider the following two cases:

Case 1: The process $N(t)$ in the risk model (1) is a Poisson process of order k. Then the risk model (1) is called a Poisson process of order k risk model.

Case 2: The process $N(t)$ in the risk model (1) is an I-Delaporte process. Then the risk model (1) is called I-Delaporte risk model.

In the first case if the insurance policies are separated in independent groups, then the number of groups has a Poisson distribution and they are homogeneous and identically distributed. The number of policies in each of the groups has a discrete uniform distribution over k points. In the second case the successive claims are of two types, such that the first type of claims are counted by the INB process and the counting process of the second type of claims is the Pólya–Aepply process. The interest is in counting all the claims in total. Then the number of claims has an I-Delaporte distribution. Let $\tau(u) = \inf\{t > 0 : X(t) < -u\}$ with the convention that $\inf \varnothing = \infty$ be the time to ruin of an insurance company, having initial capital $u \geq 0$. The ruin probability is denoted by $\psi(u) = P(\tau(u) < \infty)$ and the non-ruin probability by $\phi(u) = 1 - \psi(u)$. The relative safety loading

θ of the insurance company is defined by $\theta = \frac{E(X(t))}{E(\sum_{i=1}^{N(t)} Z_i)}$. We suppose that the safety loading is positive. The main in the application for these two models is to analyse the joint probability distribution $G(u,y)$ of the time to ruin τ and the deficit at the time of ruin $D = |u + X(\tau)|$. The function $G(u,y)$ is given by $G(u,y) = P(\tau(u) < \infty, D \le y),\ \ y \ge 0,$ see [3]. It is clear that $\lim_{y\to\infty} G(u,y) = \psi(u).$

For the Poisson process of order k risk model the function $G(u,y)$ is given by:

$$G(u,y) = (1-k\lambda h)G(u+ch,y) + \lambda h \sum_{i=1}^{k}\left[\int_0^{u+ch} G(u+ch-x,y)dF^{*i}(x) + (F^{*i}(u+ch+y) - F^{*i}(u+ch))\right] +$$

$+o(h)$. The probability distribution function of the aggregated claims is given by: $H(x) = \frac{\lambda}{k}\sum_{i=1}^{k} F^{*i}(x),$ where $H(0)=0$ and $H(\infty)=\lambda$ i.e $H(x)$ is a defective distribution function, see [4]. The function $G(u,y)$ without initial capital is given by: $G(0,y) = \frac{k\lambda}{c}\int_0^\infty [H_1(u+y) - H_1(u)]du.$

Theorem: For $u \ge 0,$ the ruin probability $\psi(u)$ satisfies the equation: $\frac{\partial\psi(u)}{\partial u} = \frac{k\lambda}{c}\left[\psi(u) - \int_0^u \psi(u-x)dH_1(x) - [1 - H_1(u)]\right],$ where $H_1(x) = \frac{H(x)}{\lambda}$ is the proper distribution function of the aggregated claims. The ruin probability with no initial capital is given by $\psi(0) = \frac{k(k+1)\lambda\mu}{2c}$. The theorem's proof is given in [4].

The nonruin probability $\phi(u)$ satisfies the equation: $\frac{\partial\phi(u)}{\partial u} = \frac{k\lambda}{c}\left[\phi(u) - \int_0^u \phi(u-x)dH_1(x)\right].$

For the I-Delaporte risk model the function $G(u,y)$ is given by:

$$G(u,y) = \left[1 - \left(\alpha + \frac{r}{\beta+t}\right)h\right]G(u+ch,y) + (1-\rho)\sum_{k=1}^{\infty}\left[\alpha\rho^{k-1} + \frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^2\left(1-(1-\rho)\frac{\beta}{\beta+t}\right)^{k-1}\right]h$$

$$\times\left[\int_0^{u+ch} G(u+ch-x,y)dF^{*k}(x) + (F^{*k}(u+ch+y) - F^{*k}(u+ch))\right],$$ where $F^{*k}(x),$ i=1,2,…is the distribution function of $Z_1 + Z_2 + \ldots + Z_k.$

The probability distribution function of the aggregated claims is given by: $H(x) = \frac{1-\rho}{\alpha + \frac{r}{\beta+t}}\sum_{k=1}^{\infty}\left[\alpha\rho^{k-1} + \frac{r}{\beta}\left(\frac{\beta}{\beta+t}\right)^2\left(1-(1-\rho)\frac{\beta}{\beta+t}\right)^{k-1}\right]F^{*k}(x),$ where $H(0)=0$ and $H(\infty)=1$ i.e. $H(x)$ is a proper distribution function, see [5]. The function $G(u,y)$ without initial capital is given by $G(0,y) = \left(\alpha + \frac{r}{\beta+t}\right)\frac{1}{c}\int_0^y [1 - H(u)]du.$

Theorem: For $u \ge 0,$ the ruin probability $\psi(u)$ for I-Delaporte risk model satisfies the equation: $\frac{\partial\psi(u)}{\partial u} = \left(\alpha + \frac{r}{\beta+t}\right)\frac{1}{c}\left[\psi(u) - \int_0^u \psi(u-x)dH(x) - [1 - H(u)]\right]$. The ruin probability with no initial capital is given by $\psi(0) = \frac{\mu}{c(1-\rho)}\left(\alpha + \frac{r}{\beta}\right)$. The theorem's proof is given in [5].

The nonruin probability $\phi(u)$ satisfies the equation: $\frac{\partial\phi(u)}{\partial u} = \left(\alpha + \frac{r}{\beta + t}\right)\frac{1}{c}\left[\phi(u) - \int_0^u \phi(u-x)dH(x)\right]$.

4. EXPONENTIALLY DISTRIBUTED CLAIMS

For the both models we consider a case of exponentially distributed claim sizes i.e $F(x) = 1 - e^{-\frac{x}{\mu}}$, $x \geq 0$, $\mu > 0$. In the first model the function $H_1(x)$ is a mixture of Erlang distribution functions: $H_1(x) = \frac{1}{k}\sum_{i=1}^{k} F^{*i}(x)$, where $F^{*k}(x)$ is the distribution function of $Erlang(i,\mu)$ distributed random variables. The corresponding density function is given by: $h_1(x) = \frac{1}{\mu k}\sum_{i=1}^{k}\frac{\left(\frac{x}{\mu}\right)^{i-1}}{(i-1)!}e^{-\frac{x}{\mu}}$, $x>0$. The survival function is: $\overline{H_1}(x) = \frac{1}{k}\sum_{i=1}^{k}\frac{\Gamma\left(\frac{x}{\mu}, i\right)}{\Gamma(i)}$, $x>0$.

The result for the function $G(0, y)$ for this model is: $G(0,y) = \frac{k\lambda}{c}\int_0^{\infty}[H_1(u+y) - H_1(u)]du$.

In the second model the distribution function $H(x)$ is: $H(x) = 1 - \frac{\alpha}{\alpha + \frac{r}{\beta+t}}e^{-\frac{1-\rho}{\mu}x} - \frac{\frac{r}{\beta+t}}{\alpha + \frac{r}{\beta+t}}e^{-\frac{1-\rho}{\mu}\frac{\beta}{\beta+t}x}$.

The result for the function $G(0, y)$ is: $G(0,y) = \frac{\mu}{(1-\rho)c}\left[\alpha\left(1 - e^{-\frac{1-\rho}{\mu}y}\right) + \frac{r}{\beta}\left(1 - e^{-\frac{1-\rho}{\mu}\cdot\frac{\beta}{\beta+t}y}\right)\right]$.

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