Algorithmic generation of isoclinism classes for 4-generator groups of nilpotency class 2

Ivo Michailov, Ivan Ivanov, Nikola Ziapkov

Abstract: *The purpose of this paper is to present a computer algorithm that classifies, with exactness to isoclinism, 4-generator groups of class 2. We prove a theoretical result that is used to construct the algorithm by using the package HAP for GAP [1].*
Key words: *GAP, HAP, nilpotency class 2.*

1. INTRODUCTION

One of the most significant results in algebra is the theorem for the unique decomposition of abelian groups as direct product of cyclic groups. Clearly, a group G is abelian if and only if $G=Z(G)$, i.e., G coincides with its center. With regard to Noether's problem and Bogomolov multipliers, Saltman [4] pointed out the need for classification of nilpotent groups of class 2. Recall that a non-abelian group $G$is of nilpotency class 2 if and only if $G'\le Z(G)$, i.e., the commutator group $G'=[G,G]$ is contained in the center $Z(G)$. As far as we know, until now only the 2-generator groups of class 2 were classified in [1]. The purpose of this paper is to present a computer algorithm that classifies, with exactness to isoclinism, 4-generator groups of class 2.

2. DESCRIPTION OF 4-GENERATOR *p*-GROUPS OF NILPOTENCY CLASS 2

Let Gbe a p-group of nilpotency class ≤ 2, let B be an abelian normal subgroup with two generators, and let the quotient group G/B be an abelian group with two generators. We can write Gas an internal product BA, where

$B=\langle \beta_1,\beta_2:\beta_1^{p^{b_1}}=\beta_2^{p^{b_2}}=1,[\beta_1,\beta_2]=1\rangle$ and $A=\langle \alpha_1,\alpha_2:\alpha_1^{p^{a_1}},\alpha_2^{p^{a_2}},[\alpha_1,\alpha_2]\in B\rangle$

for some positive integers a_1,a_2,b_1,b_2. Clearly, $B\simeq C_{p^{b_1}}\times C_{p^{b_2}}$ and $G/H\simeq C_{p^{a_1}}\times C_{p^{a_2}}$.

Since $G'\le Z(G)$, we can write $[\beta_i,\alpha_j]=\beta_1^{u_{ij}}\beta_2^{v_{ij}}\in Z(G)$ for some $0\le u_{ij}\le p^{b_1}-1,\ \ 0\le v_{ij}\le p^{b_2}-1$. We also have $[\alpha_1,\alpha_2]=\beta_1^{c_1}\beta_2^{c_2}\in Z(G)$ for some $0\le c_1\le p^{b_1}-1,\ 0\le c_2\le p^{b_2}-1$.

The first goals of this paper is to find necessary and sufficient conditions for the parameters u_{ij},v_{ij},c_1,c_2 so that G is correctly defined p-group of nilpotency class 2. We need to assure the correctness of the defining relations in G, and that G has a nilpotency class ≤ 2. Since $G'\le Z(G)$ from the well-known commutator identity $[a,bc]=[a,c][a,b][[a,b],c]$ it follows that $[a,bc]=[a,b][a,c]$ and $[ab,c]=[a,c][b,c]$. In particular $[a,b^n]=[a,b]^n$ and $[a^n,b]=[a,b]^n$ for any integer n.

2.1. Correctness. We have $1=[\beta_i,\alpha_j^{p^{a_j}}]=[\beta_i,\alpha_j]^{p^{a_j}}=\beta_1^{u_{ij}p^{a_j}}\beta_2^{v_{ij}p^{a_j}}$. Therefore,

$$u_{ij}p^{a_j}\equiv 0\left(\operatorname{mod} p^{b_1}\right),\ \ v_{ij}p^{a_j}\equiv 0\left(\operatorname{mod} p^{b_2}\right) \qquad (0.1)$$

for all i,j. Next, $1=[\beta_i^{p^{b_i}},\alpha_j]=[\beta_i,\alpha_j]^{p^{b_i}}=\beta_1^{u_{ij}p^{b_i}}\beta_2^{v_{ij}p^{b_i}}$. Therefore,

$$u_{ij} p^{b_i} \equiv 0 \left(\bmod p^{b_1}\right), \; v_{ij} p^{b_i} \equiv 0 \left(\bmod p^{b_2}\right) \qquad (0.2)$$

for all i, j. Since $\alpha_1^{p^{a_1}}, \alpha_2^{p^{a_2}} \in B$, we can write $\alpha_i^{p^{a_i}} = \beta_1^{d_i} \beta_2^{e_i}$ for $0 \le d_i \le p^{b_1} - 1, 0 \le e_i \le p^{b_2} - 1, 1 \le d_i \le 2$. Then

$$[\alpha_1^{p^{a_1}}, \alpha_2] = [\beta_1^{d_1} \beta_2^{e_1}, \alpha_2] = \beta_1^{u_{12} d_1} \beta_2^{v_{12} d_1} \beta_1^{u_{22} e_1} \beta_2^{v_{22} e_1} =$$

$$= \beta_1^{u_{12} d_1 + u_{22} e_1} \beta_2^{v_{12} d_1 + v_{22} e_1} = [\alpha_1, \alpha_2]^{p^{a_1}} = \beta_1^{c_1 p^{a_1}} \beta_2^{c_2 p^{a_1}}.$$

Therefore,

$$u_{12} d_1 + u_{22} e_1 \equiv c_1 p^{a_1} \left(\bmod p^{b_1}\right), \; v_{12} d_1 + v_{22} e_1 \equiv c_2 p^{a_1} \left(\bmod p^{b_2}\right). \qquad (0.3)$$

Similarly,

$$[\alpha_1, \alpha_2^{p^{a_2}}] = [\alpha_1, \beta_1^{d_2} \beta_2^{e_2}] = [\beta_1^{-d_2} \beta_2^{-e_2}, \alpha_1] = \beta_1^{-u_{11} d_2} \beta_2^{-v_{11} d_2} \beta_1^{-u_{21} e_2} \beta_2^{-v_{21} e_2} =$$

$$= \beta_1^{-u_{11} d_2 - u_{21} e_2} \beta_2^{-v_{11} d_2 - v_{21} e_2} = [\alpha_1, \alpha_2]^{p^{a_2}} = \beta_1^{c_1 p^{a_2}} \beta_2^{c_2 p^{a_2}}.$$

Therefore,

$$-u_{11} d_2 - u_{21} e_2 \equiv c_1 p^{a_2} \left(\bmod p^{b_1}\right), \; -v_{11} d_2 - v_{21} e_2 \equiv c_2 p^{a_2} \left(\bmod p^{b_2}\right). \qquad (0.4)$$

2.2. Nilpotency. Since $[\beta_i, \alpha_j] = \beta_1^{u_{ij}} \beta_2^{v_{ij}} \in Z(G)$, we must have

$$1 = [\beta_1^{u_{ij}} \beta_2^{v_{ij}}, \alpha_k] = \beta_1^{u_{ij} u_{1k}} \beta_2^{u_{ij} v_{1k}} \beta_1^{v_{ij} u_{2k}} \beta_2^{v_{ij} v_{2k}} = \beta_1^{u_{ij} u_{1k} + v_{ij} u_{2k}} \beta_2^{u_{ij} v_{1k} + v_{ij} v_{2k}}.$$

Therefore,

$$u_{ij} u_{1k} + v_{ij} u_{2k} \equiv 0 \left(\bmod p^{b_1}\right), \; u_{ij} v_{1k} + v_{ij} v_{2k} \equiv 0 \left(\bmod p^{b_2}\right), \qquad (0.5)$$

for any i, j, k. Similarly, from $[\alpha_1, \alpha_2] = \beta_1^{c_1} \beta_2^{c_2} \in Z(G)$, we must have

$$1 = [\beta_1^{c_1} \beta_2^{c_2}, \alpha_k] = \beta_1^{c_1 u_{1k}} \beta_2^{c_1 v_{1k}} \beta_1^{c_2 u_{2k}} \beta_2^{c_2 v_{2k}} = \beta_1^{c_1 u_{1k} + c_2 u_{2k}} \beta_2^{c_1 v_{1k} + c_2 v_{2k}}.$$

Therefore,

$$c_1 u_{1k} + c_2 u_{2k} \equiv 0 \left(\bmod p^{b_1}\right), \; c_1 v_{1k} + c_2 v_{2k} \equiv 0 \left(\bmod p^{b_2}\right), \qquad (0.6)$$

for any i, j, k.

In this way, we obtain the following.

Theorem 2.1. *The relations*

$$\beta_1^{p^{b_1}} = \beta_2^{p^{b_2}} = 1, \; [\beta_1, \beta_2] = 1, \; [\beta_i, \alpha_j] = \beta_1^{u_{ij}} \beta_2^{v_{ij}}, \; [\alpha_1, \alpha_2] = \beta_1^{c_1} \beta_2^{c_2}, \; \alpha_i^{p^{a_i}} = \beta_1^{d_i} \beta_2^{e_i}$$

define a p-group G of nilpotency class ≤ 2 if and only if the conditions (1.1)-(1.6) are satisfied.

3. GAP ALGORITHM

With regard to the Bogomolov multipliers of these groups, we are interested in the isoclinism classes of these groups. Recall that two groups G and H are called isoclinic if there exist isomorphisms $\theta : G / Z(G) \to H / Z(H)$ and $\phi : G' \to H'$ such that $\phi([\alpha, \beta]) = [\alpha', \beta']$, where $\alpha' Z(H) = \theta(\alpha Z(G))$ and $\beta' Z(H) = \theta(\beta Z(G))$. We will use the computer program GAP and the package HAP [2], to describe the algorithm for generating the isoclinism classes of the groups from Section 1. For any p, a_1, a_2, b_1, b_2 \defined by the user, the algorithm returns a representative of each class, described uniquely by the sequence of the powers $[c_1, c_2, d_1, d_2, e_1, e_2, u_{11}, u_{12}, u_{21}, u_{22}, v_{11}, v_{12}, v_{21}, v_{22}]$. Here we present our algorithm:

```
# The user must execute first the command 'LoadPackage("HAP");'
#The user must define p, a1, a2, b1, b2.
L:=[];; M:=[];; m:=0;; ID:=[];;
for u11 in [0..p^b1-1] do    for u12 in [0..p^b1-1] do    for u21 in [0..p^b1-1] do
  for u22 in [0..p^b1-1] do      for v11 in [0..p^b2-1] do    for v12 in [0..p^b2-1] do
     for v21 in [0..p^b2-1] do     for v22 in [0..p^b2-1] do
         if (a*p^a1 mod p^b1=0) and (b*p^a1 mod p^b2=0)
           and (u11*p^a1 mod p^b1=0) and (u12*p^a2 mod p^b1=0) #(1.1)
           and (u21*p^a1 mod p^b1=0) and (u22*p^a2 mod p^b1=0) #(1.1)
           and (v11*p^a1 mod p^b2=0) and (v12*p^a2 mod p^b2=0)# (1.1)
           and (v21*p^a1 mod p^b2=0) and (v22*p^a2 mod p^b2=0)# (1.1)
           and (u11*p^b1 mod p^b1=0) and (u12*p^b1 mod p^b1=0) ) #(1.2)
           and (u21*p^b2 mod p^b1=0) and (u22*p^b2 mod p^b1=0) #(1.2)
           and (v11*p^b1 mod p^b2=0) and (v12*p^b1 mod p^b2=0) #(1.2)
           and (v21*p^b2 mod p^b2=0) and (v22*p^b2 mod p^b2=0) ) #(1.2)
           and (u12*d1+u22*e1-c1*p^a1 mod p^b1=0) ) #(1.3)
           and (v12*d1+v22*e1-c2*p^a1 mod p^b2=0) #(1.3)
           and (-u11*d2-u21*e2-c1*p^a2 mod p^b1=0) ) #(1.4)
           and (-v11*d2-v21*e2-c2*p^a2 mod p^b2=0) #(1.4)
           and (u11*u11+v11*u21 mod p^b1=0) and (u11*u12+v11*u22 mod p^b1=0) #(1.5)
           and (u12*u11+v12*u21 mod p^b1=0) and (u12*u12+v12*u22 mod p^b1=0) #(1.5)
           and (u21*u11+v21*u21 mod p^b1=0) and (u21*u12+v21*u22 mod p^b1=0) #(1.5)
           and (u22*u11+v22*u21 mod p^b1=0) and (u22*u12+v22*u22 mod p^b1=0) #(1.5)
           and (u11*v11+v11*v21 mod p^b2=0) and (u11*v12+v11*v22 mod p^b2=0) #(1.5)
           and (u12*v11+v12*v21 mod p^b2=0) and (u12*v12+v12*v22 mod p^b2=0) #(1.5)
           and (u21*v11+v21*v21 mod p^b2=0) and (u21*v12+v21*v22 mod p^b2=0) #(1.5)
           and (u22*v11+v22*v21 mod p^b2=0) and (u22*v12+v22*v22 mod p^b2=0) #(1.5)
           and (c1*u11+c2*u21 mod p^b1=0) and (c1*u12+c2*u22 mod p^b1=0) #(1.6)
           and (c1*v11+c2*v21 mod p^b2=0) and (c1*v12+c2*v22 mod p^b2=0) #(1.6)
                then m:=m+1;
                F := FreeGroup(IsSyllableWordsFamily,"x1","x2","y1","y2");;
                x1 := F.1;; x2 := F.2;; y1 := F.3;; y2 := F.4;;
                rels := [x1^(p^a1)/(y1^d1*y2^e1), x2^(p^a2)/y1^d2*y2^e2, y1^(p^b1),
                y2^(p^b2), Comm(x1,x2)/(y1^c1*y2^c2), Comm(y1,x1)/(y1^u11*y2^v11),
                Comm(y1,x2)/(y1^u12*y2^v12), Comm(y2,x1)/(y1^u21*y2^v21),
                Comm(y2,x2)/(y1^u22*y2^v22)];
                H := F / rels;
                G := PcGroupFpGroup(H);
                if IsPcGroup(G) then G := PcGroupToPcpGroup(G); fi;
                Add(L,G); Add(M, [c1,c2,d1,d2,e1,e2,u11,u12,u21,u22,v11,v12,v21,v22]);
                GG:=Pcgs(G); Code:=CodePcgs(GG); ID[m] := Code;
         fi;
 od;  od;  od;  od; od;  od; od; od;
C:=IsoclinismClasses(L);;
for c in C do
pc:=Pcgs(c[1]);
        for i in [1..m] do
         if ID[i]=CodePcgs(pc) then Print(M[i]);fi;
        od;
od;
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#######

For example, if $p=3, a_1=a_2=b_1=b_2=1$, we obtain the following.

Theorem 3.1. Let G be a 3-group of nilpotency class ≤ 2, let B and the quotient group G/B be isomorphic to $C_3 \times C_3$. Then G is isoclinic to precisely one of the following two groups: $\{1\}$ and the Heisenberg group denoted by James [3] as $\Phi_2(111)$.

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ABOUT THE AUTHORS:

Prof. D. Sc. Ivo Michailov Michailov, e-mail: ivo@fmi.shu-bg.net