On Some Varieties of Algebras Defined by Low Degree Identities

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Abstract: *Some subalgebras of the $n \times n$ matrix algebra over the Grassmann algebra are investigated and low degree identities for these algebras are discussed. A special trace property is found giving an easy proof of the identities satisfied.*

Key words**:** *identities, varieties of algebras, Grassmann algebra, matrix algebras with Grassmann entries, standard polynomial*

INTRODUCTION

In several papers [7,8,9] the author investigated the PI-properties of some matrix algebras with Grassmann entries. We recall the definition of the infinite dimensional Grassmann algebra E as

$$E = E(V) = K\langle e_1, e_2, \dots \mid e_i e_j + e_j e_i = 0 i, j = 1,2,\dots\rangle,$$

where the field K has characteristic zero.

The algebra E is in focus of recent research in PI-theory. Its importance is connected with the structure theory for the T- ideals of identities of associative algebras developed by Kemer [4]. For some other applications of E one could see [8].

The significance of considering the matrix algebra $M_n(E)$ is confirmed by the following statement as the trivial isomorphism $E \otimes M_n(K) \cong M_n(E)$ holds:

Proposition 1 [3, Corollary 8.2.4]: *For every PI-algebra R there exists a positive n such that $T(R) \supseteq T(M_n(E))$, i.e. R satisfies all polynomial identities of the $n \times n$ matrix algebra $M_n(E)$ with entries from the Grassmann algebra.*

Many of the PI-properties of E and $M_n(E)$ could be found in [2,5]. Here we formulate:

Proposition 2 [5, Corollary, p. 437]: *The T-ideal $Id(E)$ is generated by the identity* $[x_1, x_2, x_3] = [x_1, x_2]x_3 - x_3[x_1, x_2] = 0$.

Proposition 3 [2, Lemma 6.1]: *The algebra E satisfies $S_n(x_1, \dots, x_n)^k = 0$ for all $n, k \geq 2$ and $S_n(x_1, \dots, x_n) = \sum_{\sigma \in Sym(n)} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$* being the standard identity.

Proposition 4 [2, Corollary 6.6]: *The algebra $M_n(E)$ does not satisfy the identity*

$$S_m^n(X_1, ..., X_m) = 0 \text{ for any } m.$$

It is an open question [1, p.356] to describe the identities of minimal degree for $M_n(E)$. Even for $n = 2$ we know very little. There is a result of Vishne [10] that the minimal degree of an identity for $M_2(E)$ is 8 and he gives the explicit form of two multilinear polynomials being identities in the algebra. Their definition is given in [10].

In the general case a result of Popov [6] states that the matrix algebra $M_n(E)$ has no identities of degree $4n - 2$.

In the paper we investigate some varieties of algebras defined by low degree identities and give examples of subalgebras of the $n \times n$ matrix algebra over the Grassmann algebra belonging to the corresponding varieties. A special trace property is found giving an easy proof for the stated results.

LAYOUT

THE VARIETY $\mathfrak{R}_1$ DEFINED BY THE IDENTITY $[x_1, x_2, x_3]x_4 = 0$

Straightforward consequences of the considered identity lead to

Proposition 5: *The elements of* $\mathfrak{R}_1$ *satisfy the identities*

$$[x_1, x_2]x_3x_4 = x_3[x_1, x_2]x_4;\ [x_1, x_2][x_1, x_3]x_4 = 0;$$

$$S_2(x_1, x_2)S_3(y_1, y_2, y_3) = 0;\ S_3(y_1, y_2, y_3)S_2^2(x_1, x_2) = 0;\ S_3^2(x_1, x_2, x_3) = 0.$$

Proof: The first identity is just another form of $[x_1, x_2, x_3]x_4 = 0$.

For proving that $[x, y, z]B = 0$ leads to the identity $[x, y][y, z]B = 0$ we use the trivial identities $[xy, y] = [x, y]y$ and $[xy, z] = x[y, z] + [x, z]y$. Thus

$$0 = [[xy, y], z]B = [[x, y]y, z]B = [x, y][y, z]B + [x, y, z]yB = [x, y][y, z]B.$$

The third identity follows from the second one and the presentation

$$S_3(x_1, x_2, x_3) = S_2(x_1, x_2)x_3 + S_2(x_2, x_3)x_1 + S_2(x_3, x_1)x_2.$$

Analogously using $x_5[x_1, x_2][x_1, x_3]x_4 = 0$ we get $S_3(y_1, y_2, y_3)S_2^2(x_1, x_2) = 0$.

Using the above representation of $S_3(x_1, x_2, x_3)$ we get that all summands in $S_3^2(x_1, x_2, x_3)$ are of type $S_2(x_i, x_j)x_u S_2(x_i, x_k)x_v$, where the indices take different values from the set $\{1,2,3\}$. Due to $[x_1, x_2]x_3x_4 = x_3[x_1, x_2]x_4$ the summands of $S_3^2(x_1, x_2, x_3)$ could be written as $x_u S_2(x_i, x_j)S_2(x_i, x_k)x_v$ and the identity $[x, y][y, z]B = 0$ gives $S_3^2(x_1, x_2, x_3) = 0$.

Now we define an algebra, different from the algebra E, from the variety $\mathfrak{R}_1$.

Let $\alpha_2, ..., \alpha_n$ be fixed elements of the field K. We consider the n-th dimensional matrix algebra $AM_n(E)$ of the matrices of type $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha_2 x_1 & \alpha_2 x_2 & \cdots & \alpha_2 x_n \\ ... & ... & ... & ... \\ \alpha_n x_1 & \alpha_n x_2 & \cdots & \alpha_n x_n \end{pmatrix}$.

Theorem 1: *The algebra* $AM_n(E)$ *belongs to the variety* $\mathfrak{R}_1$.

For proving the theorem we use the following

Proposition 6: *For any* $A, B, C \in AM_n(E)$ *the identity* $Tr[A, B, C] = 0$ *holds for* TrA *being the trace of the matrix* A.

Proof: Let $A = (a_{ij})$ for $a_{1j} = x_j, a_{ij} = \alpha_i x_j, 1 < i \le n,\ j = 1, ..., n$ and $B = (b_{ij})$ for $b_{1j} = y_j, b_{ij} = \alpha_i y_j, 1 < i \le n,\ j = 1, ..., n$. Then

$$\begin{aligned} Tr(AB) &= x_1y_1 + x_2\alpha_2y_1 + \cdots + x_n\alpha_ny_1 + \alpha_2x_1y_2 + \alpha_2x_2\alpha_2y_2 + \cdots + \alpha_2x_n\alpha_ny_2 \\ &+ \alpha_3x_1y_3 + \alpha_3x_2\alpha_2y_3 + \cdots + \alpha_3x_n\alpha_ny_3 + \cdots + \alpha_nx_1y_n + \alpha_nx_2\alpha_2y_n + \cdots + \alpha_nx_n\alpha_ny_n \\ &= x_1(y_1 + \alpha_2y_2 + ... + \alpha_ny_n) + \alpha_2x_2(y_1 + \alpha_2y_2 + ... + \alpha_ny_n) + \cdots \\ &+ \alpha_nx_n(y_1 + \alpha_2y_2 + \cdots + \alpha_ny_n) \\ &= x_1TrB + \alpha_2x_2TrB + \cdots + \alpha_nx_nTrB = TrA\,TrB. \end{aligned}$$

Thus we get $Tr[A, B] = [TrA, TrB]$ and $Tr[A, B, C] = [TrA, TrB, TrC] = 0$.

Proof of Theorem 1: Let $[X_1, X_2, X_3] = (a_{ij})$, $X_4 = (b_{ij})$ and $[X_1, X_2, X_3]X_4 = (c_{ij})$. Using the notation from Proposition 6 for the corresponding entries (a_{ij}) and (b_{ij}) we have

$$\begin{aligned} c_{ks} &= \alpha_k x_1 y_s + \alpha_k x_2 \alpha_2 y_s + \cdots + \alpha_k x_n \alpha_n y_s \\ &= \alpha_k (x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) y_s = \alpha_k Tr[X_1, X_2, X_3] y_s = 0. \end{aligned}$$

Applying Proposition 5 and Theorem 1 we get partial cases when Proposition 4 does not hold, namely

Proposition 7: For the matrix algebra $AM_2(E)$ there exists $m = 3$ such that the identity $S_m^2(X_1,...,X_m) = 0$ holds. For the matrix algebra $AM_3(E)$ there exists $m = 2$ such that the identity $S_m^3(X_1,...,X_m) = 0$ holds.

THE VARIETY $\Re_2$ DEFINED BY THE IDENTITY $x_4[x_1, x_2, x_3] = 0$

Proposition 8: *The elements of $\Re_2$ satisfy the identities*

$$\begin{aligned} & x_4[x_1, x_2]x_3 = x_4 x_3 [x_1, x_2] \\ & x_4[x_1, x_2][x_1, x_3] = 0 \\ & S_3(y_1, y_2, y_3) S_2(x_1, x_2) = 0 \\ & S_2^2(x_1, x_2) S_3(y_1, y_2, y_3) = 0 \\ & S_2^3(x_1, x_2) = 0, \quad S_3^2(x_1, x_2, x_3) = 0 \end{aligned} .$$

In [9] we consider the X - figural algebra of matrices $A_{n\times n} = (a_{ij})$ over the Grassmann algebra with nonzero elements only on the two diagonals such that $a_{ii} = a_{i,n-i+1}$ for $i = 1,...,n$. It is easily seen that $Tr[X_1, X_2, X_3] = 0$ for X_1, X_2, X_3 being from the X - figural algebra and $X_4[X_1, X_2, X_3] = 0$. Thus we have

Proposition 9: The X - figural algebra belongs to the variety $\Re_2$.

Theorem 4 in [9] gives that the T-ideal of this algebra is generated by $X_4[X_1, X_2, X_3] = 0$. As the T-ideal of $M_2(E)$ is contained in the T-ideal generated by $X_4[X_1, X_2, X_3] = 0$ we get

Proposition 10: Any algebra from the variety $\Re_2$ satisfies the identities of $M_2(E)$.

Now we consider the following generalization in the even case of the X - figural algebra, namely the $2n^2$-th dimensional matrix algebra $DM_{2n}(E)$ of the matrices of type

$$\begin{pmatrix} a_1 & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1,2n-1} & a_1 \\ 0 & a_2 & a_{23} & \cdots & \cdots & a_{2,2n-2} & a_2 & 0 \\ ... & ... & ... & ... & .. & ... & ... & ... \\ 0 & \cdots & 0 & a_n & a_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{n+1} & a_{n+1} & 0 & \cdots & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & a_{2n-1} & a_{2n-1,3} & \cdots & \cdots & a_{2n-1,2n-2} & a_{2n-1} & 0 \\ a_{2n} & a_{2n,2} & \cdots & \cdots & \cdots & \cdots & a_{2n,2n-1} & a_{2n} \end{pmatrix}.$$

The motivation is connected with the importance of the algebra of the uppertriangular matrices both in the classical case of a field and over the algebra E as well [3]. The $n \times n$ building blocks of a matrix in $DM_{2n}(E)$ include an uppertriangular matrix, its right vertical symmetry, a lowertriangular matrix and its left vertical symmetry.

Theorem 2: *The algebra* $DM_{2n}(E)$ *satisfies the identity*

$$X_{11}[X_{12}, X_{13}, X_{14}]X_{21}[X_{22}, X_{23}, X_{24}]...X_{n1}[X_{n2}, X_{n3}, X_{n4}] = 0 .$$

Proof: Let $X_{12} = (a_{ij})$, $X_{13} = (b_{ij})$ and $X_{14} = (c_{ij})$ be from $DM_{2n}(E)$ and $[X_{12}, X_{13}, X_{14}] = (m_{ij})$. Modulo the Grassmann identity we get

$$m_{ii} + m_{2n-i+1,i} = [[a_i, b_{2n-i+1}] + [a_{2n-i+1}, b_i], c_i + c_{2n-i+1}] = 0, i = 1,...,n-1$$

$$m_n = [a_n, b_n, c_n] = 0, m_{n+1} = [a_{n+1}, b_{n+1}, c_{n+1}] = 0$$

This gives that each matrix $X_{i1}[X_{i2}, X_{i3}, X_{i4}]$ has zero entries on both its diagonals. The multiplication of two such matrices has additionally zero $(n-1)$-th and $(n+2)$-th rows. The next multiplication leads to zero entries of the $(n-2)$-th and $(n+3)$-th rows as well. Thus the n-th multiplication will result into a zero matrix.

Proposition 11: *In* $DM_{2n}(E)$ *the following identities hold*:

$$S_2^{3n}(X_1, X_2) = 0$$

$$(S_3^2(X_1, X_2, X_3)S_2^2(Y_1, Y_2))^n = 0$$

$$(S_3(X_1, X_2, X_3)S_2(Y_1, Y_2))^k, \quad n = 2k$$

$$(S_3(X_1, X_2, X_3)S_2(Y_1, Y_2))^k S_3(X_1, X_2, X_3)S_2(Y_1, Y_2) = 0, \quad n = 2k+1.$$

THE VARIETY $\mathfrak{R}_3$ DEFINED BY THE IDENTITY $[x_1, x_2, x_3]x_4[x_5, x_6, x_7] = 0$

Proposition 12: *The elements of* $\mathfrak{R}_3$ *satisfy the identities*

$$[x_1, x_2][x_1, x_3]x_4[x_5, x_6][x_5, x_7] = 0$$

$$S_2(x_1, x_2)S_3(y_1, y_2, y_3)S_2^2(x_3, x_4) = 0 .$$

$$S_2^2(x_1, x_2)S_3(y_1, y_2, y_3)S_2(x_3, x_4) = 0$$

Now we consider the $(4n+1)$-th dimensional matrix algebra $CM_{2n+1}(E)$ of the matrices of type

$$\begin{pmatrix} a_{11} & 0 & ... & 0 \\ a_{21} & 0 & ... & 0 \\ ... & ... & ... & ... \\ a_{n1} & 0 & ... & 0 \\ a_{n+1,1} & a_{n+1,2} & ... & a_{n+1,2n+1} \\ a_{n+2,1} & 0 & ... & 0 \\ a_{n+3,1} & 0 & ... & 0 \\ ... & ... & ... & ... \\ a_{2n+1,1} & 0 & ... & 0 \end{pmatrix}.$$

Theorem 3: *The matrix algebra* $CM_{2n+1}(E)$ *belongs to the variety* $\mathfrak{R}_3$.

Proof: Let $A,B,C,D \in CM_{2n+1}(E)$. For $[A,B,C]=(m_{ij})$ we have $m_{11}=m_{n+1,n+1}=0$. In $[A,B,C]D=(s_{ij})$ we get $s_{n+1,2}=s_{n+1,3}=...=s_{n+1,2n+1}=0$. Thus $[A,B,C]D[A_1,B_1,C_1]=0$.

We point that if we change the places of the $(n+1)$-th row of $CM_{2n+1}(E)$ and of any of the other ones we get representatives of $2n$ more classes of algebras all of which belong to $\mathfrak{R}_3$. There are too many other $(4n+1)$-th dimensional subalgebras of $M_{2n+1}(E)$ belonging to the variety $\mathfrak{R}_3$ as well.

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REFERENCES

[1]. Belov A., L. Rowen. Computational Aspects of Polynomial Identities. Research Notes in Mathematics. Volume 9. A K Peters, 2005.

[2]. Berele A., A. Regev. Exponential growth for codimensions of some P.I. algebras. J. Algebra. Volume 241, 2001, 118-145

[3]. Giambruno A., M. Zaicev. Polynomial Identities and Assymptotic Methods. Math. Surveys and Monographs. Volume 122. American Mathematical Society, 2005.

[4]. Kemer A.R. Ideals of Identities of Associative Algebras. Trans. Math. Monogr. Volume 87. American Mathematical Society, 1991.

[5]. Krakowski D., A. Regev. The polynomial identities of the Grassmann algebra. Trans. Amer. Math. Soc. Volume 181, 1973, 429-438.

[6]. Popov A. On the minimal degree identities of the matrices over the Grassmann algebra. American University of Blagoevgrad, preprint, 1997.

[7]. Rashkova Ts. Identities of $M_2(E)$ are identities for classes of subalgebras of $M_n(E)$ as well. Proceedings of Union of Scientists – Ruse. b. 5. Volume 10, 2013, 7-13.

[8]. Rashkova Ts. On the nilpotency in matrix algebras with Grassmann entries. Serdica Math. J. Volume 38, 2012, 79-90.

[9]. Rashkova Ts. The T - ideal of the X - figural algebra. Proceedings of Union of Scientists – Ruse. book 5. Volume 11, 2014, 7-13.

[10]. Vishne U. Polynomial identities of $M_2(G)$. Commun. in Algebra. Volume 30(1), 2002, 443-454.

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