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CLASSIFICATION OF THE p-GROUPS G HAVING A NORMAL ABELIAN SUBGROUP H OF INDEX p SUCH THAT $G_{(p)} = \{1\}$ [1]

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Abstract: *In this article we prove a classification theorem for p-groups G having a normal abelian subgroup H of index p under the assumption that the p-th lower central subgroup* $G_{(p)}$ *is trivial.*

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INTRODUCTION

This paper is a sequel to (Michailov, I., Ivanov, I. (2018)). Let us first introduce some notations. The cyclic group of order n we denote by C_n. The subgroups $G_{(0)} = G$ and $G_{(i)} = [G, G_{(i-1)}]$ for $i \geq 1$ are called the lower central series of G. Let $G = \langle H, \alpha \rangle$ be a *p*-group, where H is a normal abelian subgroup of G and $\alpha^p \in H$. Denote $H^p = \{h^p : h \in H\}$ and $H(p) = \{h \in H : h^p = 1, h \notin H^p\}$. For any $\beta \in H$ define $N_G(\beta) = \langle \alpha^{-x} \beta \alpha^x : 0 \leq x \leq p-1 \rangle$, the normalizer of $\langle \beta \rangle$ in G.

Bender (Bender, H. A. (1927-1928)) determined some interesting properties of these groups. Although these results help to understand the structure of these groups, the classification problem still remains open. To the authors is not known any other attempt at that direction for nearly a century. Our goal is to make a complete description of all possible group types when the *p*-th lower central subgroup $G_{(p)}$ is trivial. Namely, we will provide the decomposition in direct normal factors together with the commutator rules. This description will be used to study Noether's problem in future works of the authors. For further reading regarding classification problems of *p*-groups, the reader is referred to (Berkovich, Y. (2008), Berkovich, Y., J. Zvonimir (2008), Berkovich, Y., J. Zvonimir (2011)).

EXPOSITION

In the following Theorem we find a decomposition of an arbitrary abelian group H as a direct product of subgroups that are 'almost' normal in G.

Theorem 1. Let p be prime and let G be a group of order p^n for $n \geq 2$ with an abelian normal subgroup H of order p^{n-1}. Choose any $\alpha \in G$ such that α generates G/H, i.e., $\alpha \notin H, \alpha^p \in H$.

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Assume also that the p-th lower central subgroup $G_{(p)}$ is trivial. Then there exist elements $\alpha_1,\dots,\alpha_s \in H$ such that the following conditions are satisfied:

i. $H=\langle\alpha_1\rangle\times\cdots\times\langle\alpha_s\rangle\times\cdots\times\langle\alpha_r\rangle$ for some $r\ge s$, i.e., $\alpha_1,\dots,\alpha_s$ are generators of direct cyclic factors of H; Let $\mathrm{ord}(\alpha_j)=p^{a_j}$ for $1\le j\le r$.

ii. $H=N_G(\alpha_1)\cdots N_G(\alpha_s)$, i.e., H is a product (not necessarily direct) of the normalizers of $\alpha_1,\dots,\alpha_s$.

iii. For any $i:1\le i\le s$ if α_i is in $Z(G)$, then $N_G(\alpha_i)=\langle\alpha_i\rangle$; if α_i is not in $Z(G)$ then there exists a natural number $k_i\le p$ and generators $\alpha_{i1},\dots,\alpha_{ik_i}\in N_G(\alpha_i)$, such that $\alpha_{i1}=\alpha_i, [\alpha_{ij},\alpha]=\alpha_{ij+1}$ for $1\le j\le k_i-1, \alpha_{ik_i}$ is in the centre $Z(G)$. Moreover, if α_{i2} is not in $Z(G)$, then $\langle\alpha_{i2},\dots,\alpha_{ik_i-1}\rangle\le H(p)$.

iv. For any $j:1\le j\le s$ such that $\alpha_{j2}\in H(p)$, put $\mathrm{H}_j=\langle\alpha_{j2},\dots,\alpha_{jk_j}\rangle$, if $\alpha_{jk_j}\in H(p)$; $\mathrm{H}_j=\langle\alpha_{j2},\dots,\alpha_{jk_j-1}\rangle$, if $\alpha_{jk_j}\notin H(p)$; $\mathcal{H}_j=\langle\alpha_j,\mathrm{H}_j\rangle$. For any $j:1\le j\le s$ such that $\alpha_{j2}\notin H(p)$, put $\mathcal{H}_j=\langle\alpha_j\rangle$. Then $H\cong\mathcal{H}_1\times\cdots\times\mathcal{H}_s$. Moreover, if $\alpha_{jk_j}\in H(p)$ then $\mathcal{H}_j=N_G(\alpha_j)\cong C_{p^{a_j}}\times(C_p)^{k_j-1}$; if $\alpha_{jk_j}\notin H(p)$ then $\mathcal{H}_j\cong C_{p^{a_j}}\times(C_p)^{k_j-2}$.

Proof. If $H^p=\{1\}$, i.e., H is a direct product of cyclic groups of order p we can apply [Michailov, I., Ivanov, I. (2018), Theorem 2]. Henceforth, we assume that there is a generator of H of order p^a for $a>1$. If $H(p)=\varnothing$, i.e., any generator α_i is of order p^{a_i} for $a_i>1$, then clearly $\mathcal{H}_j=\langle\alpha_j\rangle$ for $1\le j\le r$ and all conditions are satisfied. Henceforth, we assume also that there is a generator of H of order p.

Step I. In this Step we are going to show that there exist $\alpha_1,\dots,\alpha_s\in H$ such that conditions (i)-(iii) are satisfied. First, let us decompose H as a direct product of cyclic groups: $H=\langle\alpha_1\rangle\times\cdots\times\langle\alpha_r\rangle$. If all generators of H are central in G, then G is abelian and our theorem is obvious. Therefore, we will assume that there exists at least one generator which is not central. For any $i:1\le i\le r$ such that α_i is not central in G, define $\alpha_{i1}=\alpha_i$. Since $G_{(p)}=\{1\}$, there exist $\alpha_{i2},\dots,\alpha_{ik_i}\in H$ for some $k_i:2\le k_i\le p$ such that $[\alpha_{ij},\alpha]=\alpha_{ij+1}$, where $1\le j\le k_i-1$ and $\alpha_{ik_i}\ne 1$ is central. Since α^p is in H, from the well known formula

$$\alpha_{i1}=\alpha^{-p}\alpha_{i1}\alpha^p=\alpha_{i1}\alpha_{i2}^{\binom{p}{1}}\alpha_{i3}^{\binom{p}{2}}\cdots\alpha_{ip}^{\binom{p}{p-1}}\alpha_{ip+1},$$

where we put $\alpha_{ik_i+1}=\cdots=\alpha_{ip+1}=1$, it follows that

$$\alpha_{i2}^{\binom{p}{1}}\alpha_{i3}^{\binom{p}{2}}\cdots\alpha_{ik_i}^{\binom{p}{k-1}}=1.$$

Hence $(\alpha_{i2}\cdot\prod_{j\ne 2}\alpha_{ij}^{b_j})^p=1$ for some integers b_j. This identity clearly is impossible if the order of α_2 is greater than p. Moreover, we have $\alpha^{-1}\alpha_{i1}^p\alpha=\alpha_{i1}^p\alpha_{i2}^p=\alpha_{i1}^p$, so $\alpha^{-1}\alpha_j^p\alpha=\alpha_j^p$ for all j, i.e., $H^p\le Z(G)$. According to [Michailov, I., Ivanov, I. (2018), Lemma 1] we have that $N_G(\alpha_i)=\langle\alpha_{i1},\dots,\alpha_{ik_i}\rangle$. From $H^p\le Z(G)$ it follows also that if α_{i2} is not in $Z(G)$, then $\langle\alpha_{i2},\dots,\alpha_{ik_i-1}\rangle\le H(p)$; if $\alpha_{i2}=1$, i.e., α_{i1} is in $Z(G)$, then $N_G(\alpha_i)=\langle\alpha_i\rangle$.

Among all generators of H we can choose a subset of generators that have the maximal length k_i of their commutator chain. If there are more than one generator in this chosen set, we choose a generator with the smallest order. Denote this generator by α_1. Thus, we have that $k_1\ge k_i$ for all $i:1\le i\le r$ and $\mathrm{ord}(\alpha_1)\le\mathrm{ord}(\alpha_i)$ for all i such that $k_i=k_1$. Similarly, if $H\ne N_G(\alpha_1)$, we may choose generators $\alpha_i \in H$, $N_G(\alpha_1)\cdots N_G(\alpha_{i-1})$, for $2 \le i \le s \le r$ (after eventual renumbering), such that $H = N_G(\alpha_1)N_G(\alpha_2)\cdots N_G(\alpha_s)$. Therefore, conditions (ii) and (iii) are satisfied.

Step II. In this Step we are going to show that there exist new generators $\alpha_2',\ldots,\alpha_s' \in H$ such that $\mathrm{H}_1 \cap (\mathrm{H}_2' \cdots \mathrm{H}_s') = \{1\}$, where H_i' corresponds to H_i for α_i'. In the light of the above considerations, we may assume that $k_1 \ge k_2 \ge \cdots \ge k_s \ge 1$ and if $k_i = k_j$ for some $i < j$, then $\mathrm{ord}(\alpha_i) \le \mathrm{ord}(\alpha_j)$. We can also assume that α_2 is not central, otherwise all the remaining generators are also central, so $H = H_1 \times H_2 \times \cdots \times H_s$ and we are done.

Observe that the elements $\alpha_{i2}, \alpha_{i3}, \ldots \in \mathrm{H}_i$ are independent generators of $N_G(\alpha_i)$ for $1 \le i \le s$. Indeed, if we assume that $\alpha_{ik} = \alpha_i^{b_i p^{a_i - 1}} \prod_{1<j<k} \alpha_{ij}^{x_{ij}}$ for $0 \le b_i \le p-1$ and some $x_{ij} \ne 0$, we will get an endless commutator chain, which is impossible, G being nilpotent. However, it is possible that $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2k_2}$ are dependent modulo $N_G(\alpha_1)$, i.e., $\prod_{j=1}^{k_2} \alpha_{2j}^{x_j} \in N_G(\alpha_1)$, $\{1\}$ for $x_i : 0 \le x_i \le p-1$. If we suppose that $x_{j_1} \ne 0$ and $x_{j_0} \ne 0$ for some $1 \le j_0 < j_1 \le k_2$, we will obtain an endless commutator chain. Thus the only possibility is that there exists $\ell_2 \le k_2$ such that $\alpha_{2\ell_2} \in N_G(\alpha_1)$, $\{1\}$ and if $\ell_2 > 1$ then $\alpha_{21}, \ldots, \alpha_{2\ell_2 - 1} \notin N_G(\alpha_1)$, $\{1\}$.

We can write $\alpha_{2\ell_2} = \beta\gamma$, where $\beta \in \mathrm{H}_1$ and $\gamma \in \langle \alpha_1^{p^{a_1-1}} \rangle$. According to [5, Lemma 1], β appears in a commutator chain starting with a generator α_1' such that $N_G(\alpha_1) = N_G(\alpha_1')$, i.e., there exist $\alpha_{11}', \ldots, \alpha_{1\ell_1}' \in N_G(\alpha_1')$, such that $\alpha_{11}' = \alpha_1', [\alpha_{1j}', \alpha] = \alpha_{1j+1}'$ for $1 \le j \le \ell_1 - 1$ and $\alpha_{1\ell_1}' = \beta$ for some $\ell_1 \le k_1$. Notice that $k_1 - \ell_1 = k_2 - \ell_2$, because after β the two commutator chains coincide. Since we assumed that $k_1 \ge k_2$, we get $\ell_1 \ge \ell_2$. Define $\alpha_2' = \alpha_{1\ell_1 - \ell_2 + 1}'^{-1} \alpha_2$ and $\alpha_{21}' = \alpha_2', [\alpha_{2j}', \alpha] = \alpha_{2j+1}'$ for $1 \le j \le \ell_2 - 1$. Notice that if $\ell_1 = \ell_2$ then $\alpha_2' = \alpha_1^{-1}\alpha_2$ is a valid change since we assumed that $\mathrm{ord}(\alpha_1) \le \mathrm{ord}(\alpha_2)$. Therefore, $\alpha_{2\ell_2}' = \gamma$ and if we put $\mathrm{H}_2' = N_G(\alpha_2') \cap H(p)$, then $\mathrm{H}_1 \cap \mathrm{H}_2' = \{1\}$.

If there exists another generator, say $\alpha_3 \notin N_G(\alpha_1)N_G(\alpha_2')$, we may proceed in a similar manner. Namely, suppose that for some $t \ge 2$ there exist generators $\alpha_2', \ldots, \alpha_t'$ such that $\alpha_{i1}' = \alpha_i', [\alpha_{ij}', \alpha] = \alpha_{ij+1}'$ and $\{\alpha_{ij}' : 1 \le j \le \ell_i - 1, 2 \le i \le t\}$ are generators of H with the property that for $\mathrm{H}_i' = N_G(\alpha_i') \cap H(p)$, $2 \le i \le t$, then $\mathrm{H}_1 \cap (\mathrm{H}_2' \cdots \mathrm{H}_t') = \{1\}$. We proved this assertion for $t = 2$, and we will show that it holds for $t+1$.

Assume that $\alpha_{t+1} \notin N_G(\alpha_1)N_G(\alpha_2')\cdots N_G(\alpha_t')$ and $\prod_{i=2}^{t}\prod_{j=1}^{\ell_i - 1} \alpha_{ij}'^{x_{ij}} \cdot \prod_{j=1}^{k_{t+1}} \alpha_{t+1j}^{x_{t+1j}} \in N_G(\alpha_1)$, $\{1\}$. If we suppose that $x_{t+1j_0} \ne 0$ and $x_{t+1j_1} \ne 0$ for some $1 \le j_0 < j_1 \le k_{t+1}$ we will obtain a contradiction with the nilpotency of G. Thus the only possibility is that there exists $\ell_{t+1} \le k_{t+1}$ such that $\alpha_{t+1\ell_{t+1}} = \beta\gamma \prod_{i=2}^{t}\prod_{j=1}^{\ell_i - 1} \alpha_{ij}'^{y_{ij}}$, where $\beta \in \mathrm{H}_1, \gamma \in \langle \alpha_1^{p^{a_1-1}} \rangle$ and if $\ell_{t+1} > 1$ then the elements $\alpha_{t+11}, \ldots, \alpha_{t+1\ell_{t+1}-1}$ are not in $N_G(\alpha_1)N_G(\alpha_2')\cdots N_G(\alpha_t')$.

According to (Michailov, I., Ivanov, I. (2018), Lemma 1) (4), β appears in a commutator chain starting with a generator α_1'' such that $N_G(\alpha_1) = N_G(\alpha_1'')$, i.e., there exist $\alpha_{11}'', \ldots, \alpha_{1\ell_1'}'' \in N_G(\alpha_1'')$, such that $\alpha_{11}'' = \alpha_1'', [\alpha_{1j}'', \alpha] = \alpha_{1j+1}''$ for $1 \le j \le \ell_1' - 1$ and $\alpha_{1\ell_1'}'' = \beta$ for some $\ell_1' \le k_1$. Notice that $k_1 - \ell_1' \le k_{t+1} - \ell_{t+1}$. (Here we might have an inequality, when $\alpha_{t+1\ell_{t+1}+k_1-\ell_1'}' \notin Z(G)$.) Since we assumed that $k_1 \ge k_{t+1}$, we get $\ell_1' \ge \ell_{t+1}$. Define $\alpha_{t+1}' = \alpha_{1\ell_1' - \ell_{t+1}'' + 1}^{-1} \alpha_{t+1}$ and $\alpha_{t+11}' = \alpha_{t+1}', [\alpha_{t+1j}', \alpha] = \alpha_{t+1j+1}'$ for

$1\le j\le \ell_{t+1}-1$. Therefore, $\alpha_{t+1\ell'_{t+1}}=\prod_{i=2}^{t}\prod_{j=1}^{\ell_i-1}\alpha'^{y_{ij}}_{ij}\in N_G(\alpha_2)\cdots N_G(\alpha_t)$, so $\mathrm{H}_1\cap(\mathrm{H}'_2\cdots\mathrm{H}'_{t+1})=\{1\}$. We can continue this process until we finish the generators of H. Thus we will obtain finally that $H=N_G(\alpha_1)N_G(\alpha_2)\cdots N_G(\alpha_s)$ for some new generators denoted again by $\alpha_2,\ldots,\alpha_s$ of direct cyclic factors such that $\mathrm{H}_1\cap(\mathrm{H}_2\cdots\mathrm{H}_s)=\{1\}$.

Step III. We apply Step II consecutively for $N_G(\alpha_2)\cdots N_G(\alpha_s), N_G(\alpha_3)\cdots N_G(\alpha_s),\ldots,\ N_G(\alpha_s)$. In this way we get new generators denoted again by $\alpha_1,\ldots,\alpha_s$ of direct cyclic factors such that $\mathrm{H}_i\cap(\mathrm{H}_{i+1}\cdots\mathrm{H}_s)=\{1\}$ for $1\le i\le s-1$. This implies that

$$\mathrm{H}_i\cap\prod_{j\ne i}\mathrm{H}_j=\{1\},\quad \text{for } 1\le i\le s. \tag{1}$$

Finally, for $1\le j\le s$ such that $\alpha_{j2}\in H(p)$, put $\mathcal{H}_j=\langle\alpha_j,\mathrm{H}_j\rangle$. For any $j:1\le j\le s$ such that $\alpha_{j2}\notin H(p)$, i.e., $\alpha_{j2}\in\langle\alpha_j^{p^{a_j-1}}\rangle$, put $\mathcal{H}_j=\langle\alpha_j\rangle$. From (1) it follows that $H\cong\mathcal{H}_1\times\cdots\times\mathcal{H}_s$. Moreover, if $\alpha_{jk_j}\in H(p)$ then $\mathcal{H}_j=N_G(\alpha_j)\cong C_{p^{a_j}}\times(C_p)^{k_j-1}$; if $\alpha_{jk_j}\notin H(p)$ then $\mathcal{H}_j\cong C_{p^{a_j}}\times(C_p)^{k_j-2}$. $\square$

Finally, we are ready to prove the main classification Theorem.

Theorem 2. Let p be prime and let G be a group of order p^n for $n\ge 2$ with an abelian subgroup H of order p^{n-1}. Choose any $\alpha\in G$ such that α generates G/H, i.e., $\alpha\notin H,\alpha^p\in H$. Assume that the p-th lower central subgroup $G_{(p)}$ is trivial. Then $H=H_1\times H_2\times\cdots\times H_k$ (for some $k\ge1$) is a direct product of subgroups, where each H_i is also a direct product of subgroups: $H_i=H_{i1}\times H_{i2}\times\cdots\times H_{ij_i}$ (for some $j_i\ge1$ and for all $i=1,\ldots,k$). For any $i\ge1$ we denote by Z_i the elementary abelian p-subgroup of $H_1\times H_2\times\cdots\times H_i$ generated by the powers $\beta_\ell^{p^{b_\ell-1}}$ for all generators β_ℓ of $H_1\times H_2\times\cdots\times H_i$ of order p^{b_ℓ} (for $b_\ell>1$). Put $Z_0=\{1\}$. Then for any i, the subgroup H_i is isomorphic to one of the following six types:

I. $H_i=H_{i1}\cong C_{p^{b_1}}$ generated by an element β_1 such that $[\beta_1,\alpha]\in Z_{i-1}$, $\mathrm{ord}(\beta_1)=p^{b_1}$ for $b_1\ge1$; $H_i=N_G(\beta_1)\pmod{Z_{i-1}}$, where $N_G(\beta_1)=\langle H_{i1},[\beta_1,\alpha]\rangle$ is the normalizer of H_{i1} in G.

II. $H_1=H_{i1}\cong N_G(\beta_1)\cong C_{p^{b_1}}\times(C_p)^{k_1-1}$ for $b_1\ge1,k_1\ge2$, generated by $\beta_{11}=\beta_1,[\beta_{1j},\alpha]=\beta_{1j+1}$; $\beta_{1k_1}\in H(p)\cap Z(G)$; $\mathrm{ord}(\beta_1)=p^{b_1}$.

III. $H_i=H_{i1}\cong C_{p^{b_1}}\times(C_p)^{k_1-2}$ for $b_1\ge2,k_1\ge2$, generated by $\beta_{11}=\beta_1$ and $\beta_{1j+1}=[\beta_{1j},\alpha]$ for $1\le j\le k_1-2$; $N_G(\beta_1)=\langle H_{i1},\beta_{1k_1}\rangle$, where $\beta_{1k_1}\in Z_i$; $\mathrm{ord}(\beta_1)=p^{b_1}$; $H_i=N_G(\beta_1)\pmod{Z_{i-1}}$.

IV. $H_i=H_{i1}\times H_{i2}\times\cdots\times H_{ij_i}\cong(C_{p^{b_1}}\times(C_p)^{k_1-2})\times\cdots\times(C_{p^{b_{j_i}}}\times(C_p)^{k_{j_i}-2})$ for $t\ge2$, generated by $\beta_{i1}=\beta_i$ and $\beta_{ij+1}=[\beta_{ij},\alpha]$ for $1\le j\le k_i-2$; $N_G(\beta_1)=\langle H_{i1},\beta_{1k_1}\rangle,\ldots,N_G(\beta_{j_i})=\langle H_{ij_i},\beta_{j_ik_{j_i}}\rangle$, $\beta_{1k_1}=\beta_2^{p^{b_2-1}}\pmod{Z_{i-1}},\beta_{2k_2}=\beta_3^{p^{b_3-1}}\pmod{Z_{i-1}},\ldots,\beta_{j_i-1k_{j_i-1}}=\beta_{j_i}^{p^{b_{j_i}-1}}\pmod{Z_{i-1}},\beta_{j_ik_{j_i}}\in Z_i$; $\mathrm{ord}(\beta_i)=p^{b_i},b_i\ge2,k_i\ge2$; $H_i=N_G(\beta_1)\cdots N_G(\beta_{j_i})\pmod{Z_{i-1}}$.

V. $H_i = H_{i1} \times H_{i2} \times \cdots \times H_{ij_i} \cong (C_{p^{b_1}} \times (C_p)^{k_1-2}) \times \cdots \times (C_{p^{b_{j_i}}} \times (C_p)^{k_{j_i}-1})$ for $t \geq 2$, generated by $\beta_{i1} = \beta_i$ and $\beta_{ij+1} = [\beta_{ij}, \alpha]$ for $1 \leq j \leq k_i - 2$; $N_G(\beta_1) = \langle H_{i1}, \beta_{1k_1} \rangle, \ldots, N_G(\beta_{j_i}) = \langle H_{ij_i}, \beta_{j_i k_{j_i}} \rangle$, $\beta_{1k_1} = \beta_2^{p^{b_2-1}} \pmod{Z_{i-1}}, \beta_{2k_2} = \beta_3^{p^{b_3-1}} \pmod{Z_{i-1}}, \ldots, \beta_{j_i-1 k_{j_i-1}} = \beta_{j_i}^{p^{b_{j_i}-1}} \pmod{Z_{i-1}}$; $\beta_{j_i k_{j_i}} \in H(p) \cap Z(G), \mathrm{ord}(\beta_i) = p^{b_i}, b_i \geq 2, k_i \geq 2; H_i = N_G(\beta_1) \cdots N_G(\beta_{j_i}) \pmod{Z_{i-1}}$.

VI. $H_i = H_{i1} \times H_{i2} \times \cdots \times H_{ij_i} \cong (C_{p^{b_1}} \times (C_p)^{k_1-2}) \times \cdots \times C_{p^{b_{j_i}}}$ for $t \geq 2$, generated by $\beta_{i1} = \beta_i$ and $\beta_{ij+1} = [\beta_{ij}, \alpha]$ for $1 \leq j \leq k_i - 2$; $N_G(\beta_1) = \langle H_{i1}, \beta_{1k_1} \rangle, \ldots, N_G(\beta_{j_i}) = \langle H_{ij_i}, \beta_{j_i k_{j_i}} \rangle$, $\beta_{1k_1} = \beta_2^{p^{b_2-1}} \pmod{Z_{i-1}}, \beta_{2k_2} = \beta_3^{p^{b_3-1}} \pmod{Z_{i-1}}, \ldots, \beta_{j_i-1 k_{j_i-1}} = \beta_{j_i}^{p^{b_{j_i}-1}} \pmod{Z_{i-1}}$; $[\beta_{j_i}, \alpha] \in Z_{i-1}; b_i \geq 2, k_i \geq 2; H_i = N_G(\beta_1) \cdots N_G(\beta_{j_i}) \pmod{Z_{i-1}}$.

Proof. Let the generators $\alpha_1, \ldots, \alpha_s$ satisfy the conditions of (Michailov, I., Ivanov, I. (2018), Theorem 2). Although in the proof of (Michailov, I., Ivanov, I. (2018), Theorem 2) we assumed some ordering, regarding the commutator chains and the orders of the generators, in this proof we will not make this assumption. Since our proof will be algorithmic, we assume that for some $i \geq 1$ there exist groups $H_0 = \{1\}, H_1, \ldots, H_{i-1}$, where if $i > 1$, the groups $H_1, \ldots, H_{i-1}$ are of types I-VI, such that $H = H_0 \times \cdots \times H_{i-1} \times \mathcal{H}$, for some subgroup $\mathcal{H}$ of H. If $\mathcal{H} = \{1\}$, we are done. Now let $\mathcal{H} \neq \{1\}$. Then there exist generators, say $\alpha_1, \ldots, \alpha_t \in \mathcal{H}$.

Step 1. If α_1 is central in G modulo Z_{i-1} then $H_i = H_{i1} = \langle \alpha_1 \rangle$ is of type I. If α_1 is not central but $\langle \alpha_1 \rangle$ is normal in G modulo Z_{i-1} then $H_i = H_{i1} = \langle \alpha_1 \rangle$ is of type III (for $k_1 = 2$).
Now, let $\langle \alpha_1 \rangle$ be not normal in G modulo Z_{i-1}. According to (Michailov, I., Ivanov, I. (2018), Theorem 2) there exists a natural number $k_1 : 2 \leq k_1 \leq p$ and generators $\alpha_{11}, \ldots, \alpha_{1k_1} \in N_G(\alpha_1)$, such that $\alpha_{11} = \alpha_1, [\alpha_{1j}, \alpha] = \alpha_{1j+1}$ for $1 \leq j \leq k_1 - 1$ and α_{1k_1} is in the centre $Z(G)$. If $\alpha_{1k_1} \in H(p)$ then $H_i = H_{i1} = \langle \alpha_{11}, \ldots, \alpha_{1k_1} \rangle$ is of type II. If $\alpha_{1k_1} \in \langle \alpha_1^{p^{a_1-1}} \rangle$ modulo Z_{i-1}, where $p^{a_1} = \mathrm{ord}(\alpha_1)$, then $H_i = H_{i1} = \langle \alpha_{11}, \ldots, \alpha_{1k_1-1} \rangle$ is of type III.

Next, we suppose that $\alpha_{1k_1} \in H^p$ and $\alpha_{1k_1} \notin \langle \alpha_1^{p^{a_1-1}} \rangle$ modulo Z_{i-1}. Then $\alpha_{1k_1} = \prod_{j=1}^{t} \alpha_j^{b_{1j} p^{a_j-1}}$ modulo Z_{i-1}, where $p^{a_i} = \mathrm{ord}(\alpha_i)$, and at least one of $b_{12}, \ldots, b_{1s}$ is not 0. Let $a_{j_1} = \min\{a_j : b_{1j} \neq 0, j \geq 2\}$. If $a_{j_1} > a_1$, we put $\alpha_1' = \alpha_1^{b_{11}} \prod_{j \neq 1} \alpha_j^{b_{1j} p^{a_j - a_1}}$. Since $a_j - a_1 > 0$ for all $j \neq 1$, the product $\prod_{j \neq 1} \alpha_j^{b_{1j} p^{a_j - a_1}}$ is in the centre of G, so $\alpha_{1k_1}' = \alpha_{1k_1}^{b_{11}} = \alpha_1'^{b_{11} p^{a_1-1}}$ modulo Z_{i-1}. Therefore the group $H_i = H_{i1} = \langle \alpha_{11}', \ldots, \alpha_{1k_1-1}' \rangle$ is of type III.

We are going to consider henceforth the only remaining case: $a_{j_1} \leq a_1$. We put $\alpha_{j_1}' = \alpha_{j_1}^{b_{1j_1}} \prod_{j \neq j_1} \alpha_j^{b_{1j} p^{a_j - a_{j_1}}}$. Clearly, $\mathrm{ord}(\alpha_{j_1}') = \mathrm{ord}(\alpha_{j_1})$, so this is a legitimate change of the generator α_{j_1} (i.e., $\langle \alpha_{j_1}' \rangle$ is again a direct factor of H). Moreover, all conditions (i)-(iv) from (Michailov, I., Ivanov, I. (2018), Theorem 2) remain valid for the new set of generators $\alpha_1, \alpha_2, \ldots, \alpha_{j_1}', \ldots, \alpha_s$. We also have that $\alpha_{1k_1} = \alpha_{j_1}'^{p^{a_{j_1}-1}}$ modulo Z_{i-1}.

Step 2. We follow the pattern of Step 1 for the new generator α'_{j_1}. Since the original generators are not ordered (according to their orders), we may assume that this new generator is α_2, that is $\alpha_{1k_1} = \alpha_2^{p^{a_2-1}}$ modulo Z_{i-1}. Of course, we must keep the condition $a_2 \le a_1$ from Step 1. If α_2 is central in G modulo Z_{i-1}, then $H_i = H_{i1} \times H_{i2} = \langle \alpha_{1j} : 1 \le j \le k_1 - 1 \rangle \times \langle \alpha_2 \rangle = N_G(\alpha_1) N_G(\alpha_2)$ is of type VI. If α_2 is not central but $\alpha_{2k_2} \in \langle \alpha_1^{p^{a_1}-1}, \alpha_2^{p^{a_2-1}} \rangle$ modulo Z_{i-1}, then $H_i = H_{i1} \times H_{i2} = \langle \alpha_{1j} : 1 \le j \le k_1 - 1 \rangle \times \langle \alpha_{2\ell} : 1 \le \ell \le k_2 - 1 \rangle$ is of type IV, and $H_i = N_G(\alpha_1) N_G(\alpha_2)$ modulo Z_{i-1}. If $\alpha_{2k_2} \in H(p)$ then $H_i = \langle \alpha_{1j} : 1 \le j \le k_1 - 1 \rangle \times \langle \alpha_{2\ell} : 1 \le \ell \le k_2 \rangle$ is of type V, and $H_i = N_G(\alpha_1) N_G(\alpha_2)$ modulo Z_{i-1}.

Next, we suppose that $\alpha_{2k_2} \in H^p$ and $\alpha_{2k_2} \notin \langle \alpha_1^{p^{a_1}-1}, \alpha_2^{p^{a_2-1}} \rangle$ modulo Z_{i-1}. Then $\alpha_{2k_2} = \prod_{j=1}^{t} \alpha_j^{b_{2j} p^{a_j - 1}}$ modulo Z_{i-1}, where $p^{a_i} = \mathrm{ord}(\alpha_i)$, and at least one of $b_{23}, \ldots, b_{2s}$ is not 0. Let $a_{j_2} = \min\{a_j : b_{2j} \ne 0, j \ge 3\}$. If $a_{j_2} > a_2$, we put $\alpha'_2 = \alpha_2^{b_{22}} \prod_{j \ge 3} \alpha_j^{b_{2j} p^{a_j - a_2}}$. Since $a_j - a_2 > 0$ for all $j \ge 3$, the product $\prod_{j \ne 3} \alpha_j^{b_{2j} p^{a_j - a_2}}$ is in the centre of G, so $\alpha'_{2k_2} = \alpha_1^{b_{22} b_{21} p^{a_1 - 1}} \alpha_2'^{b_{22} p^{a_2 - 1}}$ modulo Z_{i-1}. Therefore the group $H_i = H_{i1} \times H_{i2} = \langle \alpha_{1j} : 1 \le j \le k_1 - 1 \rangle \times \langle \alpha'_{2\ell} : 1 \le \ell \le k_2 - 1 \rangle$ is of type IV, and $H_i = N_G(\alpha_1) N_G(\alpha_2)$ modulo Z_{i-1}.

We assume henceforth that $a_{j_2} \le a_2$. Since $a_{j_2} \le a_j$ for all $j \ge 3$, we may put $\alpha'_{j_2} = \alpha_{j_2}^{b_{2j_2}} \prod_{j \ne j_2, j \ge 3} \alpha_j^{b_{2j} p^{a_j - a_{j_2}}}$. Clearly, $\mathrm{ord}(\alpha'_{j_2}) = \mathrm{ord}(\alpha_{j_2})$, so this is a legitimate change of the generator α_{j_2} (i.e., $\langle \alpha'_{j_2} \rangle$ is again a direct factor of H). Moreover, all conditions (i)-(iv) from [5, Theorem 2] remain valid for the new set of generators $\alpha_1, \alpha_2, \ldots, \alpha'_{j_2}, \ldots, \alpha_s$. We also have that $\alpha_{2k_2} = \alpha_{j_2}'^{p^{a_{j_2}-1}}$ modulo Z_{i-1}.

Step 3. We apply the argument of Step 2 for the new generator α'_{j_2}. We may assume that this new generator is α_3, that is $\alpha_{2k_2} = \alpha_3^{p^{a_3-1}}$ modulo Z_{i-1}. We must keep the condition $a_3 \le a_2$ from Step 2. In all cases except one, we obtain a group H_i of type IV, V or VI. The only case which remains produces a new generator α'_{j_4} such that $\alpha_{3k_3} = \alpha_{j_4}'^{p^{a_{j_4}-1}}$ modulo Z_{i-1}. We again apply Step 2 for this new generator denoted by α_4. Since H has a finite number of generators, we see by inductive reasoning that in any case we will obtain a normal subgroup H_i of G of type IV, V or VI such that $H = H_0 \times \cdots \times H_i \times \bar{H}$ for some subgroup $\bar{H}$ of H. In this way we will obtain in the end the decomposition given in the statement of our theorem. □

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