Immersed FEM for Elliptic Interface Problems with Non-homogeneous Jump Condition of Special Type

I. Georgiev, J. Kandilarov

Abstract: *In this paper an elliptic interface problem with continuous solution and jump of the flux, that depend on the solution at the interface is considered. A numerical method, based on the Immersed Interface Finite Element Method is developed. The special jump condition in the mathematical model leads to using of a modified basis functions near the interface. Quadratic immersed finite element space is developed. Theoretical analysis are presented. Several test examples demonstrate the accuracy of the method for the linear and nonlinear case of local source and discontinuous coefficients. The results confirm third order of the numerical solution in maximum norm and second order in Sobolev norm.*
Keywords: *Immersed interface method, local jump conditions, quadratic FEM.*

1. INTRODUCTION

This paper concerns with a numerical solution of a diffusion interface model problem with point sources

$$-(\beta u_x)_x + q(x)u = f(x) - K\delta(x-\xi)g(u), \quad x \in (0,1) \tag{1}$$

$$u(0) = 0, \qquad u(1) = 0, \tag{2}$$

where δ is the Dirac delta function, concentrated on the interface ξ, $\xi \in (0,1)$. We assume that $\beta(x) > 0$ is piecewise continuous and may have finite jump on the interface. As a result of the local sources the solution is continuous, but the jump of the flux depends on the solution. Under some assumptions for the regularity of the solution the problem ((1)), ((2)) can be written in an equivalent form without delta function using the following conjugation conditions on the interface ξ:

$$[u]_{x=\xi} = u(\xi+0) - u(\xi-0) = 0, \tag{3}$$

$$[\beta u_x]_{x=\xi} = Kg(u(\xi)). \tag{4}$$

Problems of this type arise when we consider a diffusion equation with localized chemical reactions. As a result of the reactions the derivatives are discontinuous across the interfaces (local sites of reactions).

Numerous methods have been developed for interface problems, see [9], [11] and references therein. The Immersed Interface Method (IIM) proposed by LeVeque and Li [9] solves elliptic equations with jump relations, which are known functions, defined on the interface.

Some $2D$ problems with jump conditions, that depend on the solution on the interface are considered by J. Kandilarov and L. Vulkov [3], [4], [6] using finite difference schemes.

The main goal of this work is the application of the Immersed Interface Finite Element Method (IIFEM) to the proposed elliptic problem and theoretical validation of its implementation. In [2] special linear basis functions are used in order to achieve second order of convergence in maximum norm. In this paper on the base of quadratic basis functions we improve the rate of convergence of the numerical solution. The organization of the paper is as follows. In Section 2 the weak formulation of the problem is given. Next, in Section 3 the theoretical analysis for the linear local source case is done. In Section 4 numerical experiments that confirm third order of the method in maximum norm are presented.

2. WEAK FORMULATION OF THE PROBLEM

Let us introduce the Sobolev space $H^1(0,1)$, the form

$$a(u,v)=\int_0^1(\beta(x)u'(x)v'(x)+q(x)u(x)v(x))dx+Kg(u(\xi))v(\xi), \quad (5)$$
$$u(x),v(x)\in H^1(0,1)$$

and linear one

$$b(f,v)=\int_0^1 f(x)v(x)dx. \quad (6)$$

Then the weak solution of (1)-(2) can be defined as a function $u\in H^1(0,1)$ such that:

$$a(u,v)=b(f,v) \quad \forall v\in H^1(0,1), \quad (7)$$

and u satisfying the boundary conditions (2). As is proved in [7], a classical solution of (1)-(4) is also a weak solution. Using monotone methods one can prove that if $K>0$ and the function $g(u)$ is bounded on each finite interval and satisfies $0\le g_0\le g'(u)$, then the problem (7) has a unique solution $u(x)\in H^1(0,1)$.

3. DISCONTINUOUS COEFFICIENTS AND LINEAR OWN SOURCE

Let us start with the case of linear own source, i.e. in (1) $g(u)=u$ and for simplicity let $q=0$. For the numerical solution of (7) we use a uniform grid in x direction with mesh parameter h:

$$\omega_h=\{x_i=ih,\quad i=0,1,...,2N,\quad h=1/2N\}.$$

Let $\mathrm{T}_h=\bigcup_{k=0}^{N-1}e_k$ be a partition of $\overline{\Omega}=[0,1]$, where elements e_k are the intervals $e_k=[x_{2k},x_{2k+2}]$.

If ξ satisfies $x_{2k}\le\xi<x_{2k+2}$, then the element e_k is called an *interface element*. The other elements are called *noninterface elements*.

If k-th element is a noninterface element, then we use the three standard Lagrange type quadratic FE local nodal functions $\phi_{k,i}(x_k)$, $i=1,2,3$ such that

$$\phi_{k,i}(t_{k,j})=\begin{cases}1, & i=j,\\ 0, & i\ne j,\end{cases}$$

where $t_{k,j}=x_{2k+j-1}$, $j=1,2,3$.

We denote with $V^h=span\{\phi_i, i=0,1,...,2N\}$ the space of all possible linear combinations of the basis functions: $V^h=\{v_h: v_h=\sum_{i=1}^{2N}\eta_i\phi_i(x)\}$ and define its subspace $V_0^h=\{v\in V^h \mid v(0)=v(1)=0\}$. Then the discrete analogue of (7) is:

Find a function $u_h=u_h(x)\in V_0^h$ *such that*

$$a(u_h,v_h)=(f,v_h)\quad \forall v_h\in V_0^h, \quad (8)$$

and

$$u_h(0)=0,\quad u_h(1)=0. \quad (9)$$

Let the interface element be e_J. Then we seek the corresponding quadratic local nodal IFE basis function in the following form, $i=1,2,3$:

$$\phi_{2J+i-1}(x)=\phi_{J,i}(x)=\begin{cases}a_2^i x^2+a_1^i x+a_0^i, & if \quad x\le\xi,\\ b_2^i x^2+b_1^i x+b_0^i, & if \quad x>\xi.\end{cases} \qquad (10)$$

To find these modified basis functions we impose that the jump conditions (3) and (4) are satisfied. Also, an additional condition is the jump of the second derivative, which we obtain from (1):

$$\phi_{J,i}(t_{J,j})=\begin{cases}1, & j=i,\\ 0, & j\ne i,\end{cases} \quad \begin{cases}[\phi_{J,i}]_\xi=0,\\ [\beta\phi_{J,i}']_\xi=K\phi_{2J+i-1}(\xi), \quad i,j=1,2,3.\\ [(\beta\phi_{J,i}'(x))']_\xi=0,\end{cases} \qquad (11)$$

In Fig. 1 the modified quadratic basis functions are presented in the case of $\xi=1/3$, $\beta^+=1$, $\beta^-=3$, $2N=160$, $K=0.4278$.

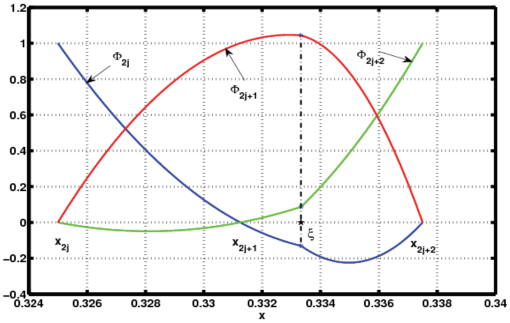


Figure 1: The interface basis functions for $\xi=1/3$, $\beta^-=3$, $\beta^+=1$, $K=0.4278$, $2N=160$.

Using the notation $\mathbf{u}_h=[u_0^h,u_1^h,...,u_{2N}^h]$, we obtain for the solution of the discrete problem the following linear system:

$$\mathbf{K}\mathbf{u}_h=\mathbf{F}, \qquad (12)$$

where

$$\mathbf{K}=\|K_{ij}\|; \quad K_{ij}=a(\phi_i(x),\phi_j(x)),$$
$$\mathbf{F}=\|F_i\|; \quad F_i=b(f(x),\phi_i).$$

The matrix $\mathbf{K}$ and $\mathbf{F}$ are known as *stiffness and force matrices*, respectively. The matrix $\mathbf{K}$ has three nonzero elements for $i=1,3,...,2N-1$ and five nonzero elements for $i=2,4,...,2N-2$.

4. ANALYSIS OF THE QUADRATIC IMMERSED INTERFACE FEM

In this section we will prove that the numerical solution obtained by the modified basis function is third order accurate in infinite norm.

In [5], [7] it is proved that if $K>0$ then the problem has a unique solution. Moreover it is proved that K may be negative, but different from some constant, that depends on $\beta^\pm$, q and ξ. Then we can define

$$\|u'''\|_\infty=\max\{|u_{xxx}^-|,|u_{xxx}^+|,\sup_{0<x<\xi}|u_{xxx}|,\sup_{\xi<x<1}|u_{xxx}|\}, \qquad (13)$$

which is bounded.

On each element e_k we define also the piecewise quadratic function $u_I(x)$

$$u_I(x) = \phi_k(x)u_{2k} + \phi_{2k+1}(x)u_{2k+1} + \phi_{2k+2}(x)u_{2k+2}, \quad x_{2k} \le x \le x_{2k+2}, \; k = 0,1,..N-1, \qquad (14)$$

that interpolates $u(x)$ at the node points.

Theorem 1 *Let $K > 0$, $\beta^- > 0$ and $\beta^+ > 0$. Then the system (11) has an unique solution.*

Proof. Let $\xi \in e_J$ and moreover $x_{2J+1} \le \xi < x_{2J+2}$. Then from (11) we obtain the following system for the coefficients a_2^i, a_1^i, a_0^i, b_2^i, b_1^i, b_0^i :

$$\begin{pmatrix} \mathrm{x}_{2\mathrm{J}}^{\;2} & \mathrm{x}_{2\mathrm{J}} & 1 & 0 & 0 & 0 \\ \mathrm{x}_{2\mathrm{J}+1}^{\;2} & \mathrm{x}_{2\mathrm{J}+1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{x}_{2\mathrm{J}+2}^{\;2} & \mathrm{x}_{2\mathrm{J}+2} & 1 \\ \xi^2 & \xi & 1 & -\xi^2 & -\xi & -1 \\ K\xi^2 + 2\beta^-\xi & \beta^- + K\xi & K & -2\beta^-\xi & -\beta^+ & 0 \\ 2\beta^- & 0 & 0 & -2\beta^+ & 0 & 0 \end{pmatrix} \begin{pmatrix} a_2^i \\ a_1^i \\ a_0^i \\ b_2^i \\ b_1^i \\ b_0^i \end{pmatrix} = \begin{pmatrix} \delta_{1,i} \\ \delta_{2,i} \\ \delta_{3,i} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (15)$$

where $\delta_{j,i}$ is the Kroneker delta.

The determinant $det(A)$ of the matrix is decomposed as a sum of two determinants $det(A) = det(A_1) + det(A_2)$, where the fifth row is decoupled. Then we have:

$$\begin{aligned} & det(A) = det(A_1) + det(A_2) \\ & = 2\beta^-\beta^+ h\big((\xi - \mathrm{x}_{2\mathrm{J}})(\xi - \mathrm{x}_{2\mathrm{J}+1}) - 2h^2\big) - 2\beta^+ h(\xi - \mathrm{x}_{2\mathrm{J}})(\xi - \mathrm{x}_{2\mathrm{J}+1}) \\ & + 2K\beta^+ h(\xi - \mathrm{x}_{2\mathrm{J}})(\xi - \mathrm{x}_{2\mathrm{J}+1})(\xi - \mathrm{x}_{2\mathrm{J}+2}). \end{aligned}$$

It easy to see, that with the assumption $K > 0$ the determinant $det(A) < 0$ for arbitrary mesh size h and hence the system (15) has an unique solution.

We have the following theorem on the accuracy of the interpolating function.

Theorem 2 *For the interpolation function u_I the inequality holds:*

$$\|u(x) - u_I(x)\|_\infty \le Ch^3 \|u'''(x)\|_\infty \qquad (16)$$

where C is a constant independent of h.

Proof. Let for simplicity $\beta^- = \beta^+ = 1$. We use Taylor expansion:

$$u(x) = \mathrm{u}(\xi(- u_x^-(\xi - x) + \frac{u_{xx}^-(\xi - x)^2}{2} + O(h^3),$$

$$u(x_{2J}) = \mathrm{u}(\xi(- u_x^-(\xi - \mathrm{x}_{2\mathrm{J}}) + \frac{u_{xx}^-(\xi - \mathrm{x}_{2\mathrm{J}})^2}{2} + O(h^3),$$

$$u(x_{2J+1}) = \mathrm{u}(\xi(- u_x^-(\xi - \mathrm{x}_{2\mathrm{J}+1}) + \frac{u_{xx}^-(\xi - \mathrm{x}_{2\mathrm{J}+1})^2}{2} + O(h^3),$$

$$u(x_{2J+2}) = \mathrm{u}(\xi(- u_x^+(\xi - \mathrm{x}_{2\mathrm{J}+2}) + \frac{u_{xx}^+(\xi - \mathrm{x}_{2\mathrm{J}+2})^2}{2} + O(h^3).$$

From $[u]_\xi = 0$, $[u_x]_\xi = Ku(\xi)$, $[u_{xx}]_\xi = [f] = 0$ we express u^+, u_x^+, u_{xx}^+ with u^-, u_x^-, u_{xx}^-. Then we have:

$$u(x) - u_I(x) = u(\xi)\left(1 - \sum_{i=0}^{2}\left(a_i^1 + a_i^2 + a_i^3(1 + K(x_{2J+2} - \xi))\right)x^i\right)$$

$$+ u_x^-(\xi)\left((x-\xi) - \sum_{i=0}^{2}\left(a_i^1(x_{2J} - \xi) + a_i^2(x_{2J+1} - \xi) + a_i^3(x_{2J+2} - \xi)\right)\right)x^i\right)$$

$$+ \frac{1}{2}u_{xx}^-(\xi)\left((x-\xi)^2 - \sum_{i=0}^{2}\left(a_i^1(x_{2J} - \xi)^2 + a_i^2(x_{2J+1} - \xi)^2 + a_i^3(x_{2J+2} - \xi)^2)\right)x^i\right)$$

$$+ O(h^3).$$

Using the system (15) we prove that all the terms in the parentheses are equal to zero and hence the theorem is truth.

Theorem 3 *Let $u(x)$ be the solution of (7) and $u_I(x)$ be defined in (14). Then for* $0 \le k \le N-1$

$$\int_{x_{2k}}^{x_{2k+2}} \left(\beta(x)(u(x) - u_I(x))' v_{h'}(x) + \delta(x-\xi)K(u(x) - u_I(x))v_h(x)\right)dx \le Ch^4, \quad \forall v_h \in V_h,$$

(17) where the constant C does not depend on h.

Following the error estimation made in [10] we have the theorem of convergence for the modified IIFEM.

Theorem 4 *Let $u_h(x)$ be the solution obtained from the FEM with the modified basis function. Then*

$$\|u(x) - u_h(x)\|_\infty \le Ch^3 \|u'''(x)\|_\infty. \tag{18}$$

5. NUMERICAL EXPERIMENTS

The model example is:

$$(\beta u_x)_x = 12x^2 + K\delta(x-\xi)u(x)$$

with an exact solution

$$u(x) = \begin{cases} \dfrac{x^4}{\beta^-} + \xi + 2, & 0 \le x \le \xi, \\ \dfrac{x^4}{\beta^+} + x + 2 + \left(\dfrac{1}{\beta^-} - \dfrac{1}{\beta^+}\right)\xi^4, & \xi \le x \le 1, \end{cases}$$

The constant K is

$$K = (\beta^- \beta^+)/(\xi^4 + \beta^- \xi + 2\beta^-).$$

We chose $\xi = 1/3$, $\beta^+ = 1$, $\beta^- = 3$ and $\beta^+ = 3$, $\beta^- = 1$.

Let us introduce the following norms in the involved Sobolev spaces:

$$\| u \|_\infty = \text{ess sup}_{x \in \Omega} |u(x)|, \quad \| u \|_0 = \left(\int_\Omega u^2 dx\right)^{1/2}, \quad \| u \|_1 = (\| u \|_0^2 + \| u' \|_0^2)^{1/2}.$$

In Table 1 the mesh refinement analysis of the error in maximum norm are presented. The third order of the method is clearly seen.

Table 1: Grid refinement analysis. The error in maximum norm $\| u - u_h \|_\infty$.

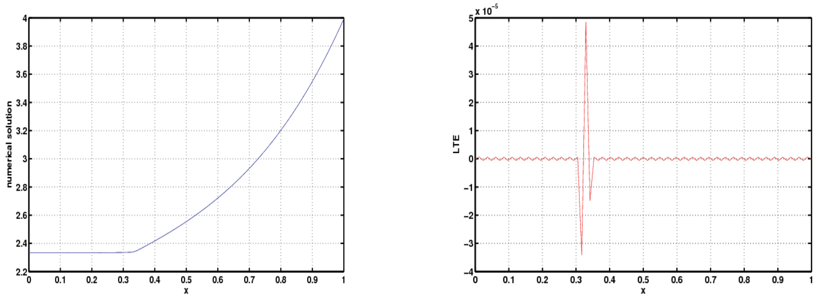
N	$\xi = 1/3,\ \beta^- = 3,\ \beta^+ = 1$		
	$\| u - u_h \|_\infty$	ratio	rate
10	6.2122e-005	-	-
20	5.6830e-006	10.9312	3.4504
40	9.3606e-007	6.0712	2.6020
80	1.0828e-007	8.6448	3.1118
160	8.1898e-009	13.2213	3.7248

In Table 2 the mesh refinement analysis in the Sobolev norm are presented. The second order of the method is confirmed.

Table 2: Grid refinement analysis. The error in Sobolev norm $\| u - u_h \|_1$.

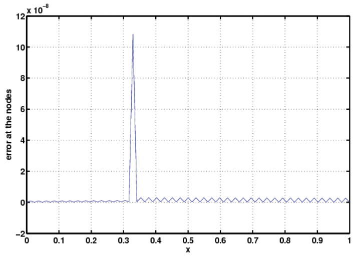
N	$\xi = 1/3,\ \beta^- = 3,\ \beta^+ = 1$		
	$\| u - u_h \|_1$	ratio	rate
10	0.0345	-	-
20	0.0073	4.7260	2.2406
40	0.0022	3.3182	1.7304
80	5.3182e-004	4.1367	2.0485
160	1.3998e-004	3.7993	1.9257

On Fig. 2a) the numerical solution u_h and on Fig. 2b) the local truncation error with $N = 40,\ \xi = 1/3,\ \beta^- = 1,\ \beta^+ = 3$ are depicted. On Fig. 3a) the error $\| u(x) - u_h(x) \|_\infty$ at the mesh points and on Fig. 3b) the error $\| u(x) - u_{I,h}(x) \|_\infty$ of the interpolated solution are shown.

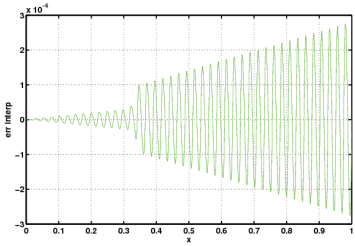


2a) 2b)

Figure 2: a) The numerical solution ; b) The local truncation error.



3a)



3b)

Figure 3: a) The error at the nodes; b) The error of the interpolated solution.

CONCLUSIONS

In this contribution we have studied the Immersed Interface Finite Element Method for the problems with discontinuous coefficients and jump conditions of the flux on the interface that depend on the solution. The standard quadratic basis functions are modified to satisfy the jump conditions. The numerical results confirm third order of the method in maximum norm.

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