A tensor approach to Heisenberg uncertainty relations

Elena Evtimova

***Abstract**: The representation of quantum energy-momentum tensor of a particle via a sequence of 4-dimensional discrete pure matter energy-momentum tensors leads to real dispersion of momentum and relativistic uncertainty relations.*

***Key words:** discrete and continual energy-momentum tensor, relativistic uncertainty relations*

INTRODUCTION

In quantum physics, the uncertainty principle is expressed as mathematical inequalities asserting a fundamental limit to the precision by which certain pairs of physical quantities of a particle, e.g. position x and momentum p, can be simultaneously known. The original argument that such a limit should exist was given by Heisenberg [1]. This interpretation is based upon the reasoning that the uncertainty inequalities arise only from the wave properties inherent in the quantum mechanical description of nature [2].

Here we shall talk over the existence of another possibility extrapolating the Shan's idea of discrete motion of quantum particles [3] to the hypothesis of 4-dimensional discrete existence of quantum particles. This can be done having in mind the representation of the energy-momentum tensor of any quantum particle through *averaging of certain sequences of sub-quantum units whose existence is discrete not only into space, but also in time* [4].

The purpose of the work is to prove that the quantum uncertainty relations follow also from the averaged energy-momentum tensor representation of these sequences of 4-dimensional discrete sub-quantum units, defined upon a specific discrete lattice over the continual Minkowski space.

DISCRETE ENERGY-MOMENTUM TENSOR OF A SCALAR PARTICLE

In special relativity theory the energy-momentum tensor (EMT) is a basic mathematical model of the objects structure. A set of localized 3D matter units is presented by summation of a set of second rank tensors of dust particles type [5]

$$T = \sum_a \int d\tau\, \delta^4(x - x(\tau))\, m\vec{\upsilon}_a \otimes \vec{\upsilon}_a \,.$$

The 4-dimesional δ-function, taken as distribution in the above tensor, is defined along the world line of the particle depending on the continuous time variable (τ) and integration takes place over a small 4D volume. Due to this fact the above tensor is dependent only on 3D variables of space and 3D velocities of different particles enumerated by the index a.

Now we introduce a standing scalar wave sequence of 4D singularities and the hypothesis is that it represents the corpuscular part of the quantum wave-particle. The notations below are as follows: a scalar function $\overset{n}{\varepsilon} = \pm 1$ defined over the set of natural numbers *N* (*n* = 1, 2, 3, …); a discrete (finite or countable) sequence $[\overset{n}{x}]$ of events in the Minkowski space-time; four-dimensional delta functions $\overset{n}{\delta}(x - \overset{n}{x})$ with poles in the events $\overset{n}{x}$ and $[\overset{n}{u}_\alpha]$, ($\alpha = 0,1,2,3; \forall\, n \in \mathrm{N}$) is a set of four-dimensional vectors in the events $\overset{n}{x}$ defined by their scalar product $\overset{n}{u}_\beta \overset{n}{u}{}^\beta = 1$. By means of these *a discrete standing wave sequence of flashing on and off singularities* represented by pure matter tensor field singularity distribution considered on the above discrete sequence of space-time events $[\overset{n}{x}]$ ($n \in N$) is introduced:

$$\tau_{\alpha\beta}(x) = hc\sum_n \overset{n}{\varepsilon}\,\delta(x-\overset{n}{x})\overset{n}{u}_\alpha \overset{n}{u}_\beta = hc\sum_n \overset{n}{\tau}_{\alpha\beta};\ \alpha,\beta = 0,1,2,3\,, \tag{2.1}$$

Here h is the Planck's quantum of action and c is the speed of light.
By means of this tensor field (2.1) we represent the *flashing on and off indivisible magnitudes or quanta* as a set of pure matter EMTs whose densities are 4D δ-functions. We assume that between all different arrangements of these discrete sets $[\overset{n}{x}]$ there exists one that is natural for a given quantum particle. We consider tensors of the type (2.1) as 4D discrete and singular micro-scale energy-momentum tensors of quantum particles. The connection between them and the wave characteristics of some quantum objects is discussed in [4, 6].

A macroscopic averaging (integration) of the above-introduced micro-scale EMT over a given space-time volume element $\Delta\Omega$ leads to tensor field of the form

$$t_{\alpha\beta}(\Delta\Omega) = \frac{hc\sum_{i=1}^{p}\overset{i}{\varepsilon}\overset{i}{u}_\alpha \overset{i}{u}_\beta}{p\,\Delta\Omega}\,. \tag{2.2}$$

Here the summation is over all of the singularities disposed into the region $\Delta\Omega$. If $\Delta\Omega$ and $\Delta\Omega'$ are two adjacent "physically infinitely small" space-time volumes, the respective tensors $t_{\alpha\beta}(\Delta\Omega)$ and $t_{\alpha\beta}(\Delta\Omega')$ discern "physically infinitely little". Thus, macroscopically $t_{\alpha\beta}$ appears as a continuous function in Minkowski space. We assume that the average value (2.2) of the tensor distribution (2.1) over any physically small region in space-time can be represented in the form of a pure matter tensor, i.e. $t_{\alpha\beta} = \rho u_\alpha u_\beta$, provided that all usual requirements of smoothness of the scalar density ρ and four-vectors u_α over a certain region of space-time are fulfilled.

CONTINUAL ENERGY-MOMENTUM TENSOR OF A SCALAR PARTICLE IN ONE-DIMENSIONAL POTENTIAL WELL

The wave function for a scalar particle in one-dimensional potential well is a solution to Klein-Gordon equation $\left(\partial_{00} - \Delta + k^2\right)\Psi = 0,$ where $\partial_{00} = \partial^2/\partial x^0 \partial x^0$, $\Delta = \vec{\nabla}\bullet\vec{\nabla}$, $\vec{\nabla}$ is the 3D gradient operator, $k = \frac{m_0 c}{\hbar}$ is the wave number and m_0 is the rest mass of the particle. This wave function differs from zero (Ψ≠0) inside a two-dimensional region that is given by $\Omega = \{x^0 \in (-\infty,+\infty),\ x \in [0,d]\}$, where d is the size of the well. We consider the simplest case of stationary solutions $\Psi(x^0, x) = \sqrt{\tfrac{2}{d}}\sin xk_x\, e^{-k_0 x^0}$. Here $k_0 = \sqrt{k^2 + k_x^2}$ is the zero component of the wave vector of the quantum particle. The boundary condition $\Psi(x^0, x \ge d) = 0$ leads to $k_x = \frac{n\pi}{d}$, $n = 1,2,3,...$.

The continual EMT of any particle is found by means of Noether's theorem [8]. The normalized to the macroscopic energy-momentum $P_0 = E = \hbar c k_0$ of a quantum particle EMT is given by

$$T_{\alpha\beta} = \frac{\hbar c}{2k_0}\left(\partial_\alpha \Psi^* \partial_\beta \Psi + \partial_\alpha \Psi \partial_\beta \Psi^*\right) - \frac{\hbar c}{2k_0}\left[\partial_\sigma \Psi^* \partial^\sigma \Psi - k^2 \Psi^* \Psi\right], \tag{3.1}$$

In the case of the above considered stationary solutions in one dimensional potential well the components of this EMT are as follows

$$T_{00}(x,k_x) = \frac{\hbar c}{k_0 d}\left(k_x^2 + 2k^2 \sin^2 xk_x\right) > 0;\ T_{01} = T_{10} = 0;\ T_{11}(x,k_x) = \frac{\hbar c}{k_0 d}k_x^2 > 0\,. \tag{3.2}$$

Consider two unit vectors $\underset{1}{u} = (u_0, u)$ and $\underset{2}{u} = (u_0, -u)$, i.e. $\underset{i}{u} \bullet \underset{i}{u} = 1, i = 1,2$, defined in the coordinate system of the tensor $T_{\alpha\beta}$ and suppose their definition region is the same as it is for $T_{\alpha\beta}$. These vectors may be used to construct two pure matter energy-momentum tensors as follows

$$\underset{1}{\bar{t}}_{\alpha\beta} = hc\begin{bmatrix} u_0u_0 & u_0u \\ u_0u & uu \end{bmatrix} \text{ and } \underset{2}{\bar{t}}_{\alpha\beta} = hc\begin{bmatrix} u_0u_0 & -u_0u \\ -u_0u & uu \end{bmatrix}. \tag{3.3}$$

Let us represent the normalized energy-momentum tensor $T_{\alpha\beta}$ by the following equation

$$T_{\alpha\beta} = \frac{\chi}{2}\left(\underset{1}{\bar{t}}_{\alpha\beta} + \underset{2}{\bar{t}}_{\alpha\beta}\right). \tag{3.4}$$

The last suggestion for the structure of EMT together with the normalization condition of the vectors $\underset{i}{u}$ to unity leads to a quadratic algebraic system of equations that is

$$hc\chi\, u_0u_0 = T_{00}, \quad hc\chi\, u\,u = T_{11}, \quad u_0^2 - u^2 = 1. \tag{3.5}$$

The solution of these equations for the scalar density function χ is

$$\chi = \frac{1}{hc}T_\sigma^\sigma = \frac{k^2}{\pi k_0 d}\sin^2 xk_x\,. \tag{3.6}$$

The components of the unit vectors are as follows

$$u_0 = \sqrt{\frac{T_{00}}{T_\sigma^\sigma}} = \sqrt{\frac{k_x^2 + 2k^2\sin^2 xk_x}{2k^2\sin^2 xk_x}} \text{ and } u = \sqrt{\frac{T_{11}}{T_\sigma^\sigma}} = \sqrt{\frac{k_x^2}{2k^2\sin^2 xk_x}}. \tag{3.7}$$

It is obvious that these components are singular functions over the borders of the definition region $x = 0$ and $x = d$.

DISPERSIONS OF AVERAGED EMT AND UNCERTAINTY RELATIONS

The representation of the quantum field EMT via a sequence of pure matter tensors

$$\chi \underset{m}{\bar{t}}_{\alpha\beta} = hc\chi \underset{m}{u}_\alpha \underset{m}{u}_\beta, \; m = 1,2 \tag{4.1}$$

leads directly to a relativistic modification of Heisenberg uncertainty relations. This will be proved by the following considerations. The differential of the momentum is

$$dP = \chi \underset{1}{\bar{t}}_{01}\, dx = \frac{\hbar c}{k_0 d} k_x\sqrt{k_x^2 + 2k^2\sin^2 xk_x}\, dx\,. \tag{4.2}$$

The total momentum in the definition interval $[0, d]$ will be

$$\Delta P = \frac{\hbar c}{k_0 d}\int_0^\pi \sqrt{k_x^2 + 2k^2\sin^2\varphi}\, d\varphi; \quad \varphi = xk_x\,. \tag{4.3}$$

As $\sin^2\varphi = 1 - \cos^2\varphi$, then the last equation becomes

$$\Delta P = \frac{\hbar c}{k_0 d}\int_0^\pi \sqrt{k_x^2 + 2k^2(1 - \cos^2\varphi)}\, d\varphi\,. \tag{4.4}$$

The last integral transforms into elliptic integral by change of variables $\varphi \to \varphi - \pi/2$ and the fact that $\cos^2\varphi = \sin^2(\varphi - \pi/2)$. Let $\alpha = \varphi - \pi/2$. Now, if $\varphi \in [0, \pi]$, then $\alpha \in [-\pi/2, +\pi/2]$. Using the relativistic relation between the wave-vectors $k_x^2 = k_0^2 - k^2$ the above integral transforms to

$$\Delta P = \frac{\hbar c}{k_0 d}\sqrt{k_0^2 + k^2}\int_{-\pi/2}^{+\pi/2} \sqrt{k_0^2 - b^2\sin^2\alpha}\, d\alpha; \quad b^2 = \frac{2k^2}{k_0^2 + k^2} < 1\,. \tag{4.5}$$

The function under the last integral is symmetric in relation to the medium $\alpha = 0$ of the integration interval, so we get

$$\Delta P = \frac{2\hbar c}{d}\sqrt{1+\frac{k^2}{k_0^2}}\,E(b,\pi/2) \tag{4.6}$$

where $E(b,\pi/2)$ is complete elliptic integral of second kind [9]. If *b* < 1, $E(b,\pi/2)$ > 1 always and if b = 1 then $E(b,\pi/2) = 1$.

Since the particle is confined in a potential well, i.e. in the region $\Delta x = d$ we conclude that

$$\Delta P \Delta x = 2\hbar c\sqrt{1+\frac{k^2}{k_0^2}}\,E(b,\pi/2)\ . \tag{4.7}$$

Because of the fact that the complete elliptic integral of second kind $E(b,\pi/2) > 1$ always it follows that the following inequality holds

$$\Delta P \Delta x > 2\hbar c\sqrt{1+\frac{k^2}{k_0^2}}\ . \tag{4.8}$$

characteristic feature of this result is that it gives us an opportunity to make an evaluation that leads tc

A. Let $k_0 \to k$, then $\sqrt{1+\frac{k^2}{k_0^2}} \underset{k_0\to k}{\to} \sqrt{2}$, $b^2 \underset{k_0\to k}{\to} 1$ and $E(1,\pi/2) = 1$. Consequently:

$$\Delta P \Delta x > 2\sqrt{2}\,\hbar c\ . \tag{4.9}$$

B. Let $k_0 \to \infty$ then $\sqrt{1+\frac{k^2}{k_0^2}} \underset{k_0\to\infty}{\to} 1$, $b^2 \underset{k_0\to\infty}{\to} 0$, $E(0,\pi/2) = \pi/2$. Consequently:

$$\Delta P \Delta x < \pi\,\hbar c\ . \tag{4.10}$$

Therefore, taking into consideration (4.9) and (4.10), it turns out that the product of the relativistic momentum and position dispersions has a lower and upper limit, i.e. can be written as

$$2\sqrt{2}\,\hbar c < \Delta P \Delta x < \pi\,\hbar c\ . \tag{4.11}$$

The conclusion following from the last inequalities is that as much as the energy of a particle (k_0 >> *k*) is bigger, so much bigger is the dispersion of momentums of the corresponding sequence of 4D singularities in the region $\Delta x = d$. This means that so much inaccurate would be the measurement of the momentum

$$\frac{2\sqrt{2}\,\hbar c}{\Delta x} < \Delta P < \frac{\pi\,\hbar c}{\Delta x}\ . \tag{4.12}$$

As one can notice from the above considerations these inaccuracies have a lower and upper limit depending on the size of the region ($\Delta x = d$) in which we are trying to localize the particle.

CONCLUSIONS

The above obtained relations (4.11) and (4.12) following from the hypothesis that the EMT of a quantum particle (in a potential well) can be decomposed into a set of discrete and singular in space-time pure matter tensors are the *relativistic uncertainty relations* between the coordinate and the corresponding component of the momentum. In contrast to the Copenhagen interpretation of quantum mechanics [10] we suggest that the wave function, together with the corresponding energy-momentum tensors, is an objective characteristic of the matter state. Hence, Δ*P* is a dispersion of the momentum that really exist in the region of size $\Delta x = d$, where the wave-particle is confined. In other words, the uncertainties Δ*P* and Δx in the inequalities (4.11) and (4.12) are conditioned by the very nature of matter and only after that from the imperfection of our measurements or measuring devices. In addition to that we found the existence of two limits to the dispersion of relativistic momentum – the left and right sides of inequalities in (4.12).

The standard Heisenberg relations in non-relativistic limit are $\Delta P \Delta x > \hbar/2$ and they are usually connected with the fact that the operators of momentum and position do not commute. Using the tensor approach of Noether's theorem does not require the explicit introduction of operators. As it is pointed out in [5] and [6] matter tensors are associated with really existing in space-time distributions of objects. Hence, the inequalities (4.11) and (4.12) can be considered as ensuing from the existence of real dispersions of some kind of discrete and singular sequences of sub-quantum units (2.1). The 'jumps' of these units over the space-time lattice $[\overset{n}{x}]$ ($n \in N$) are determined by the relativistic constants *h* and *c*. Thus, both limits in (4.11) and (4.12) show that the uncertainty which is inherent in the nature of quantum things may be considered as limited by their very nature. In terms of (4.12) the ***dispersion of momentum*** $\boldsymbol{\Delta P}$ of a particle in potential well with size d cannot be less than $2\sqrt{2}\hbar c/d$ and bigger than $\pi\hbar c/d$, i.e. it ***is limited from below and above***.

REFERENCES

[1] Heisenberg uncertainty relations, http://en.wikipedia.org/wiki/Uncertainty_principle

[2] Nave R., The Uncertainty Principle, http://hyperphysics.phy-astr.gsu.edu/hbase/uncer.html

[3] Shan G., (2000), What quantum mechanics describes is discontinuous motion, http://arxiv.org/PS_cache/physics/pdf/0001/0001012v1.pdf

[4] Evtimova E., (2012) Discrete and Continual Energy-Momentum Tensor Distributions in Cylindrical Potential Well, http://www.atiner.gr/papers/MAT2012-0258.pdf, ISSN **2241-2891**

[5] Thorne K., Part IV, General Relativity, http://www.pma.caltech.edu/Courses/ph136/yr2004/0423.1.K.pdf

[6] Bertschinger E., Number-Flux Vector and Stress-Energy Tensor, http://web.mit.edu/edbert/GR/gr2b.pdf

[7] Evtimova E., (2011) Mathematical representation of phase space atomism, http://www.atiner.gr/abstracts/2011MAT-ABST.pdf, 27

[8] Noether's theorem, http://en.wikipedia.org/wiki/Noether's_theorem

[9] Weisstein, E. W., "Elliptic Integral of the Second Kind", *MathWorld* – A Wolfram Web Resource, http://mathworld.wolfram.com/EllipticIntegraloftheSecondKind.html

[10] Copenhagen interpretation, http://en.wikipedia.org/wiki/Copenhagen_interpretation

За контакти:

Гл.ас. д-р Елена Евтимова, секция Физика, ДЕО, ул. Коста Лулчев 27, София 1111, aloniar@abv.bg

Докладът е рецензиран.