# Differential Equations for Turbulent Flow

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***Abstract**: In the current work is introduce different type of equation for motion of turbulent flows. These equations are derive differently from that of Reynolds approach. The main conclusion is the application of Boussinesq hypothesis for effective viscosity.*

***Key words:** turbulence, turbulence stresses, equations for motion of turbulent flow*

## INTRODUCTION

Reynolds equations for motion of turbulent flow are make on the basis of the hypothesis of equality of instantaneous velocity and the sum of the average at time during pulsation and component [1], [2]. A model of a turbulent flow It is make at which is need a modeling of turbulent stresses. These stresses are describe in the literature, have two-dimensional character [2], [3] and lead to a determination of a one shear stress.

Modern software products (ANSYS, Fluent) are build on the work describe in this approach. Main disadvantage in their description [4], is that they use incorrectly the phrase "Reynolds equations" which sometimes misleads the reader.

In current work are derived equations in exact form of those used in the software mention above. It is shown the possibility to determine the three-dimensional spectrum of the turbulent stresses, and hence the whole picture of the flow in a three-dimensional setting.

## MATHEMATICAL MODEL

At the conclusion of the equations for motion of turbulent fluid is starting from the equations of fluid dynamics in stresses and also the hypothesis of Boussinesq for effective viscosity is apply. Equations of fluid dynamics in streses in scalar type are as follow:

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}=\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \quad (1)$$

$$\rho\frac{\partial v}{\partial t}+\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}=\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z} \quad (2)$$

$$\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z}=\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z} \quad (3)$$

where are $\sigma_x,\sigma_y,\sigma_z$ the normal stresses and $\tau_{ij}$ are tangential stresses.

According to Newton's law $P=aS+b$ where $P$ and $S$ are tensors of internal stresses and deformation velocity and $a=2\mu$ and $b$ are tensor constants. Then for tangential stress is obtained:

$$\begin{aligned}
\tau_{yx}&=\tau_{xy}=\mu_{ef}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\
\tau_{yz}&=\tau_{zy}=\mu_{ef}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\\
\tau_{zx}&=\tau_{xz}=\mu_{ef}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)
\end{aligned} \quad (4)$$

For normal stresses for incompressible fluid the equations are as follow:

$$\sigma_x = -p + 2\mu_{ef}\frac{\partial u}{\partial x}$$
$$\sigma_y = -p + 2\mu_{ef}\frac{\partial v}{\partial y} \tag{5}$$
$$\sigma_z = -p + 2\mu_{ef}\frac{\partial w}{\partial z}$$

According to the hypothesis of Boussinesq total shear stress can be define as the sum of viscous and tangential stresses. On this basis it is assume that:

$$\tau_{eff} = \tau + \tau_T \tag{6}$$

And for the effective viscosity $\mu_{eff}$ is assuming the equation:

$$\mu_{eff} = \mu + \mu_T \tag{7}$$

Regarding (7) it is necessary to be emphasize that in contrast to the dynamic coefficient of viscosity ($\mu$), $\mu_T$ is not a constant and is not a property of the fluid, and the quality of the stream. But this expression make possible to conclude the equations of motion of a turbulent fluid.

Substitution (4,5 and 7) in equations 1-3 and then follows:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left[(\mu+\mu_T)\frac{\partial u}{\partial x}\right] + \\ + \frac{\partial}{\partial y}\left[(\mu+\mu_T)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[(\mu+\mu_T)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \tag{8}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \\ +2\frac{\partial}{\partial y}\left[(\mu+\mu_T)\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[(\mu+\mu_T)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \tag{9}$$

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] + \\ + \frac{\partial}{\partial y}\left[(\mu+\mu_T)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] + 2\frac{\partial}{\partial z}\left[(\mu+\mu_T)\frac{\partial w}{\partial z}\right] \tag{10}$$

The above equations are processing and then for the systems is obtaine:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) + 2\frac{\partial}{\partial x}\left[\mu_T\frac{\partial u}{\partial x}\right] + \\ + \frac{\partial}{\partial y}\left[\mu_T\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu_T\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \tag{11}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) + 2\frac{\partial}{\partial y}\left[\mu_T\frac{\partial v}{\partial y}\right] + \\ + \frac{\partial}{\partial x}\left[\mu_T\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu_T\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \tag{12}$$

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) + 2\frac{\partial}{\partial z}\left[\mu_T\frac{\partial w}{\partial z}\right] + \\ + \frac{\partial}{\partial x}\left[\mu_T\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu_T\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \tag{13}$$

At these equations is add and the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{14}$$

## ANALYSIS OF THE RESULTS

Equations (11) ÷ (13) which are obtain in their right side are contains the apparent turbulent normal and tangential stresses:

$$\sigma_x^{'}=\mu_T\frac{\partial u}{\partial x}=\mu_T e_x \tag{15}$$

$$\sigma_y^{'}=\mu_T\frac{\partial v}{\partial y}=\mu_T e_y \tag{16}$$

$$\sigma_z^{'}=\mu_T\frac{\partial w}{\partial z}=\mu_T e_z \tag{17}$$

$$\tau_{xy}^{'}=\tau_{yx}^{'}=\mu_T\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\mu_T e_{xy} \tag{18}$$

$$\tau_{xz}^{'}=\tau_{zx}^{'}=\mu_T\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\mu_T e_{xz} \tag{19}$$

$$\tau_{yz}^{'}=\tau_{zy}^{'}=\mu_T\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=\mu_T e_{yz} \tag{20}$$

Turbulent dynamic viscosity is a value, which depends of the coordinates $x, y, z$, and it can not be considered equivalent to the ratio of dynamic viscosity. On the other hand, turbulent stresses are depends of gradient of velocity components in each point of the stream. The receiving value for $\sigma$ and $\tau$ can be assume and that reflect the three dimensionality of the flow by influence their gradients velocity components, respectively. component of the tensor of deformation velocity:

$$S=\begin{pmatrix} e_x, e_{yx}, e_{zx} \\ e_{xy}, e_y, e_{zy} \\ e_{xz}, e_{yz}, e_z \end{pmatrix} \tag{21}$$

A similar idea is embed in the program products Ansyss and Fluent.

Since a numerical calculation of the task allows at each point to calculate the distribution of the velocity component, $u, v$ and $w$ and it is not difficult to obtain $\sigma^{'}$ and $\tau_{ij}^{'}$.

## TENSOR OF TURBULENT STRESSES

Based on the mention above it can be write the turbulent stress tensor

$$\begin{vmatrix} \sigma_x^{'}, \tau_{yx}^{'}, \tau_{xz}^{'} \\ \tau_{xy}^{'}, \sigma_y^{'}, \tau_{yz}^{'} \\ \tau_{xz}^{'}, \tau_{yz}^{'}, \sigma_z^{'} \end{vmatrix} = -\begin{vmatrix} \rho\overline{u^{'2}}, \rho\overline{u^{'}v^{'}}, \rho\overline{u^{'}w^{'}} \\ \rho\overline{u^{'}v^{'}}, \rho\overline{v^{'2}}, \rho\overline{v^{'}w^{'}} \\ \rho\overline{u^{'}w^{'}}, \rho\overline{w^{'}v^{'}}, \rho\overline{w^{'2}} \end{vmatrix} \tag{22}$$

Taking into account $(15 \div 20)$ the left side of equations (22) are replace as follows:

$$\begin{vmatrix} \mu_T \frac{\partial u}{\partial x}, \mu_T \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \mu_T \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \mu_T \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \mu_T \frac{\partial v}{\partial y}, \mu_T \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \mu_T \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \mu_T \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), \mu_T \frac{\partial w}{\partial z} \end{vmatrix} = -\begin{vmatrix} \rho\overline{u'^2}, \rho\overline{u'v'}, \rho\overline{u'w'} \\ \rho\overline{u'v'}, \rho\overline{v'^2}, \rho\overline{v'w'} \\ \rho\overline{u'w'}, \rho\overline{w'v'}, \rho\overline{w'^2} \end{vmatrix} \tag{23}$$

At this way can be determinate Reynolds stresses with their corresponding values in the left side of the equation (23). This allows the calculation of the full picture of turbulent flow - all normal and tangential turbulent stresses. It can be consider that the turbulence is determinedin a three-dimensional appearance.

## EXAMPLE OF DETERMINATION OF $\mu_T$

In the recording system equation $(11 \div 13)$ determination of $\mu_T$ can be done by using some of the well known and apply in the most popular commercial software (Fluent, ANSYSS etc.). First it can be done by model of Prandtl

$$\mu_T = \rho l \frac{\partial u}{\partial y} \tag{24}$$

where $l$ is mixing length and it is determine by the expression $l = const.b(x)$:

Other approach is to use a $k-\varepsilon$ model

$$\mu_T = \rho C_\mu \frac{k^2}{\varepsilon} \tag{25}$$

Turbulent kinetic energy $k$ and dissipation velocity $\varepsilon$ are determine by the equations

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{1}{y^j}\frac{\partial}{\partial y}\left(y^j \frac{\upsilon_t}{\sigma_k}\frac{\partial k}{\partial y}\right) - \overline{u'v'}\frac{\partial u}{\partial y} - \varepsilon \tag{26}$$

$$\rho u\frac{\partial \varepsilon}{\partial x} + \rho v\frac{\partial \varepsilon}{\partial y} = \frac{1}{y^j}\frac{\partial}{\partial y}\left[y^j \rho \frac{\upsilon_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right] + C_{\varepsilon 1}\rho\frac{\varepsilon}{k}\upsilon_T\left(\frac{\partial u}{\partial y}\right)^2 - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \tag{27}$$

where the constants are according Table1.

***Table 1***

| Constant | $C_\mu$ | $\sigma_k$ | $\sigma_\varepsilon$ | $C_{1\varepsilon}$ | $C_{2\varepsilon}$ | $C_{3\varepsilon}$ | $\sigma_T$ | $R$ |
|---|---|---|---|---|---|---|---|---|
| Value | 0,09 | 1 | 1,3 | 1,43 | 1,92 | 1,43 | 0,9 | 0,588 |
| Literature | | [3] | [1] | | [4] | | [1] | |

The approach to solve the problem can be as follows: using the mention above software products dynamic turbulent viscosity can be calculate. Further, it is necessary to make a less difficult procedure to determine the component of the tensor (23). Of course there is some uncertainty because initially $\mu_T$ is determine in a two-dimensional staging, but this is compensate with next examination. It can be consider as a second more accurate approximation of the task.

## CONCLUSION

Method of recording the equations for motion of turbulent fluid which are describe allows to model three-dimensional turbulence flows and to solve practical problems of Fluid Mechanics, Heat Transfer and others.

**LITERATURE**


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