Recovering Plane Curves by One of Their Conformal Invariants[2]

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Abstract: *The group of rotations SO(3), preserving the unit sphere $\boldsymbol{S}^2$ in the Euclidean space $\boldsymbol{R}^3$, induces via a stereographic projection π a subgroup F_0 of the Möbius group in the plane. The Euclidean theory of the Frenet plane curves is transferred naturally on the Riemann sphere $\boldsymbol{S}^2$. We find invariant of Frenet plane curves under the group F_0 which defines any Frenet plane curve up to transformation of the group F_0. We use this invariant to construct plane curves and for that purpose we apply an algorithm in the computer system Mathematica.*

Key words: *Möbius group, Frenet plane curves, stereographic projection, Riemann sphere.*

1. INTRODUCTION

The two-dimensional unit sphere $\boldsymbol{S}^2 \subset \boldsymbol{R}^3$, centered at the origin, is a model of the Gauss plane, supplied with a point of infinity ∞. This model is realized via a stereographic projection π from the north pole of the sphere onto the plane $\boldsymbol{R}^2$ through the equator. Making the point of infinity ∞ correspond to the north pole, π is a bijective conformal map in $\boldsymbol{R}^3$. Under this map the sphere $\boldsymbol{S}^2$ is known as a Riemann sphere. The group of rigid motions on the sphere $\boldsymbol{S}^2$ coincides with the group of rotations SO(3) in $\boldsymbol{R}^3$. Any spherical curve on the unit sphere, parameterized by an arc-length parameter, is defined up to a rigid motion on the sphere by a function called "spherical curvature" (see [6]). In [6] Tazawa interprets the Euclidean theory of the curves and surfaces in $\boldsymbol{R}^3$ on the three-dimensional unit sphere in $\boldsymbol{R}^4$ applying the algebra of quaternions. It is well-known that any Frenet curve is defined up to a rigid motion in the Euclidean space by its Euclidean curvatures. This fundamental theorem in the Euclidean curve theory is extended with respect to the group of direct similarities in [3]. Any Frenet plane curve, parameterized by an arc-length parameter of its spherical tangent indicatris, is defined up to a direct similarity by its shape curvature. The differential geometry of the curve with respect to the Möbius group is developed by Fialkow [4], Udo Hertrich-Jeromin [5] et al. But they don't consider the problem of recovering a curve from its conformal invariants. In the presented paper we explore pairs of curves correspondent via the stereographic projection π. We find a relation between the spherical curvature of a curve on the sphere $\boldsymbol{S}^2$ and the Euclidean curvature of the corresponding plane curve. This relation determines an invariant of a plane curve with respect to a subgroup F_0 of the Möbius group in the plane. We prove existence and uniqueness theorems for plane curves, defined up to a transformation from the conformal group F_0. These theorems give us an algorithm of recovering plane curves.

2. PRELIMINARIES

Let $\boldsymbol{S}^2 : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ be the unit sphere in $\boldsymbol{R}^3$, centered at the origin O, and $l(x^1, x^2, x^3),\ x^1 \neq 1$ be the position vector of an arbitrary point on $\boldsymbol{S}^2$, different from the pole $P(1,0,0)$. Denoting by (u^1, u^2) the coordinates of any point in the equatorial plane, the reverse stereographic map $\pi^{-1} : (u^1, u^2) \to (x^1, x^2, x^3)$ is given by the equations

$$x^1 = \frac{(u^1)^2 + (u^2)^2 - 1}{1 + (u^1)^2 + (u^2)^2}$$

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$$l: \quad x^2 = \frac{2u^1}{1+(u^1)^2+(u^2)^2}$$
$$x^3 = \frac{2u^2}{1+(u^1)^2+(u^2)^2}, \qquad (u^1, u^2) \in R^2.$$

We have the following identifications $S^2 \cong R^2 \cup \infty \cong C \cup \infty \cong P^1(C)$, where $P^1(C)$ is one-dimensional projective space with respect to the field of the complex numbers C. Then

$$\pi^{-1}: z \to \left(\frac{\|z\|^2-1}{\|z\|^2+1}, \frac{2z}{\|z\|^2+1}\right), z \in C, \pi^{-1}(\infty) = \mathrm{P},$$

where $\|z\|$ is the norm of the complex number z and

$$\pi: \ m = (t, z) \in S^2 \setminus \mathrm{P} \to \frac{1}{1-t}.z, \qquad t \in R, \qquad t \neq 1, \qquad \pi(\mathrm{P}) = \infty. \qquad (\text{see [1]})$$

Any rigid motion on the sphere S^2 induces via the stereographic projection π *a* Möbius transformation in the plane. The next theorem describes the transformation from the subgroup F_0 of the Möbius group in the plane, generated by the group of rigid motion on the sphere S^2.

Theorem 2.1. *The elements of the subgroup F_0 of the Möbius group in the plane, corresponding of the group SO(3) of rotations on the sphere S^2 via the stereographic projection π, are the following transformations*:

*a) rotations in **C**, represented by the equation* $f(z) = a.z, \ \|a\| = 1; \ a, z \in C;$

b) Möbius transformations in $C_\infty = C \cup \infty$, *represented by the equation*

$$f(z) = \frac{a.z+b}{-\overline{b}.z+\overline{a}}, \|a\|^2 + \|b\|^2 = 1; \ a, b, z \in C, b \neq 0,$$

where $\overline{a}$ and $\overline{b}$ are complex conjugates of the complex numbers a and b, respectively.

We omit the proof because it can be easily found in any complex function book.

It is clear that F_0 is a three-parameter subgroup of the Möbius group in the plane. Let $\mathrm{R}_{\mathrm{M}}^{\alpha} \in SO(3)$ be a rotation on the sphere S^2 about the axis (OM), defined by the point M with polar coordinates $(\varphi, \psi), 0 \leq \varphi \leq 2\pi, 0 \leq \psi \leq \frac{\pi}{2}$, through the angle . If M is the pole P of the stereographic projection π then $\rho_{\mathrm{O}}^{\alpha} = \pi \circ \mathrm{R}_{\mathrm{P}}^{\alpha} \circ \pi^{-1} \in F_0$ is a rotation, centered at the origin, with an equation $f(z) = e^{i\alpha}.z$. If $\mathrm{M} \neq \mathrm{P}$ then $\pi \circ \mathrm{R}_{\mathrm{M}}^{\alpha} \circ \pi^{-1} \in F_0$ is a Möbius transformations with an equation $f(z) = \frac{a.z+b}{-\overline{b}.z+\overline{a}}, \|a\|^2 + \|b\|^2 = 1; \ a, b, z \in C, b \neq 0,$ where $a = \cos\frac{\alpha}{2} + \mathrm{i}.\sin\frac{\alpha}{2}.\cos\psi, b = \sin\frac{\alpha}{2}.\sin\psi.e^{\mathrm{i}\left(\varphi-\frac{\pi}{2}\right)};$

$$\cos\frac{\alpha}{2} = \mathrm{Re}(a), \ 0 \leq \alpha \leq 2\pi, \ \varphi = \mathrm{Arg}(b) + \frac{\pi}{2}, \ \ \sin\psi = \frac{\|b\|}{\sqrt{1-\mathrm{Re}^2(a)}}, \cos\psi = \frac{\mathrm{Im}(a)}{\sqrt{1-\mathrm{Re}^2(a)}}.$$

One-parameter set of curves, equivalent to a given plane curve with respect to the group F_0, can be constructed applying the computer system Mathematica (see Fig.1).

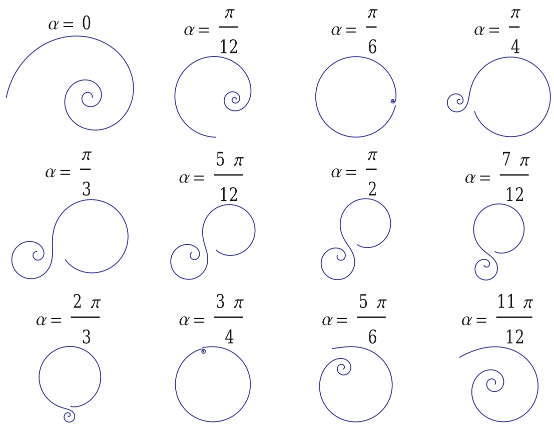

Figure1. One-parameter set of plane curves equivalent to the logarithmic spiral

3. INVARIANTS OF A PLANE CURVE WITH RESPECT TO THE GROUP F_0

We define the sphere $S^2 \subset R^3$ by the equation:

$$l(u^1,u^2) = \left\{\frac{(u^1)^2+(u^2)^2-1}{1+(u^1)^2+(u^2)^2}, \frac{2u^1}{1+(u^1)^2+(u^2)^2}, \frac{2u^2}{1+(u^1)^2+(u^2)^2}\right\}, (u^1,u^2) \in R^2.$$

The standard orthogonal tangential frame field (l_1,l_2) at any point of $S^2 \setminus P$ is given by the equations:

$$l_1 = \frac{\partial l}{\partial u^1} = \frac{4}{(1+(u^1)^2+(u^2)^2)^2}\cdot\left\{u^1, \frac{(1-(u^1)^2+(u^2)^2)}{2}, -u^1u^2\right\}$$

$$l_2 = \frac{\partial l}{\partial u^2} = \frac{4}{(1+(u^1)^2+(u^2)^2)^2}\cdot\left\{u^2, -u^1u^2, \frac{(1+(u^1)^2-(u^2)^2)}{2}\right\}.$$

Let (e_1,e_2) be the usual orthonormal frame field in R^2. We denote by $<\ ,\ >$ the standard dot product in the Euclidean plane R^2 and replace the expression $\frac{4}{(1+(u^1)^2+(u^2)^2)^2} = \frac{4}{(1+|u|^2)^2}$ in (1), where $|u| = \sqrt{u^2} = \sqrt{<u,u>}$ is the length of the position vector $u = (u^1,u^2)$ in the plane, by μ. The differential π_*^{-1} of the map π^{-1} is determined by $\pi_*^{-1}e_i = l_i$, $i = 1,2$. Then $l_1^2 = l_2^2 = \mu$ and the components of the metric tensor *g* on the sphere S^2 are

$$g_{ij} = \begin{cases} 0, & i \neq j \\ \mu, & i = j \end{cases}.$$

Hence the Christoffel symbols on S^2 are represented by the equalities

$$\Gamma_{ij}^k = \Gamma_{ji}^k = \begin{cases} -\sqrt{\mu}\,.u^j, & k = i; \\ \sqrt{\mu}\,.u^k, & k \neq i = j. \end{cases}$$

We consider a Frenet plane curve $c: u(s) = (u^1(s), u^2(s))$, parameterized by an arc-length parameter s, and denote by a prime " ′ " the differentiation with respect to the parameter s.

Lemma 3.1. Let $v(s) = \left(\sum_{i,j}\Gamma_{ij}^1.u^{i'}.u^{j'}, \sum_{i,j}\Gamma_{ij}^2.u^{i'}.u^{j'}\right)$, $i,j = 1,2$ be a vector function in the plane, defined by the curve *c*. Then $v = \sqrt{\mu}\ .u - \mu.\left(\frac{1}{\sqrt{\mu}}\right)'.u'$.

Proof. Applying the equalities (2) we get

$\sum_{i,j}\Gamma^1_{ij}.u^{i'}.u^{j'} = \Gamma^1_{11}.\left(u^{1'}\right)^2 + 2.\Gamma^1_{12}.u^{1'}.u^{2'} + \Gamma^1_{22}.\left(u^{2'}\right)^2 =$
$-\sqrt{\mu}.u^1.\left(u^{1'}\right)^2 - 2.\sqrt{\mu}.u^2.u^{1'}.u^{2'} + \sqrt{\mu}.u^1.\left(u^{2'}\right)^2 = -\sqrt{\mu}.u^{1'}.2.\left(u^1.u^{1'} + u^2.u^{2'}\right) +$
$+\sqrt{\mu}.u^1.\left(\left(u^{1'}\right)^2 + \left(u^{2'}\right)^2\right) = -\sqrt{\mu}.u^{1'}.< u,u >' + \sqrt{\mu}.u^1.\overbrace{|u'|^2}^{1} = -\sqrt{\mu}.(u^2)'.u^{1'} + \sqrt{\mu}.u^1.$

Similarly, $\sum_{i,j}\Gamma^2_{ij}.u^{i'}.u^{j'} = -\sqrt{\mu}.(u^2)'.u^{2'} + \sqrt{\mu}.u^2.$

Since $(u^2)' = |u|^{2'} = \left(\frac{2}{\sqrt{\mu}} - 1\right)' = \sqrt{\mu}.\left(\frac{1}{\mu}\right)'$, the proof is completed. □

Let γ be the corresponding spherical curve on the sphere S^2, obtained via the reverse stereographic projection π^{-1} from the plane curve c. We determine γ by the vector function $l = l(\sigma)$, where σ is an arc-length parameter of γ, and denote by ∇ the covariant differentiation on S^2. The unit tangent vector t of a curve γ is $t(\sigma) = l'$. For the Frenet frame ltn at an arbitrary point of γ we have the equalities:

$$t = l', \qquad t' = l + \tilde{\kappa}.n, \qquad n' = \tilde{\kappa}.t .$$

The function $\tilde{\kappa}(s) = |\nabla_t t|$ is called a spherical curvature of the curve γ (see [6]).

Theorem 3.1. *Let $c: u(s) = (u^1(s), u^2(s))$ be a Frenet plane curve, parameterized by an arc-length parameter s, and γ be its spherical pre-image on the sphere S^2 via the stereographic projection π. If κ and $\tilde{\kappa}$ are the Euclidean curvature and the spherical curvature of the curves c and γ, respectively, then*

$$\tilde{\kappa}^2(s) = \frac{1}{\mu}.\kappa^2(s) - 1 - \frac{2}{\mu}.(S_\sigma)(s),$$

where $\sigma = \sigma(s)$ is an arc-length function of γ and $(S_\sigma)(s)$ is the Schwarzian derivative of $\sigma = \sigma(s)$.

Remark. The Schwarzian derivative of an arbitrary function $f = f(t)$ is expressed as $(S_f)(t) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}.\left(\frac{f''}{f'}\right)^2$, where we denote by " ' " the derivative with respect to the parameter *t*.

Proof. Let $\gamma: l = l(\sigma)$. Then $\dot{l} = \frac{dl}{ds} = u^{1'}.l_1 + u^{2'}.l_2 = u^{i'}.l_i$, using an Einstein summation. Since $d\sigma = \sqrt{\mu}.ds$ is the linear element of γ on S^2, we have that $t = \frac{dl}{d\sigma} = \frac{dl}{ds}.\frac{ds}{d\sigma} = \frac{1}{\sqrt{\mu}}.\dot{l}$. Applying a covariant differentiation on the sphere S^2 we obtain that

$\nabla_t t = \nabla_{\frac{1}{\sqrt{\mu}}\dot{l}}\frac{1}{\sqrt{\mu}}.\dot{l} = \frac{1}{\sqrt{\mu}}.\left(\frac{1}{\sqrt{\mu}}.\nabla_{\dot{l}}\dot{l} + \dot{l}\left(\frac{1}{\sqrt{\mu}}\right).\dot{l}\right) = \frac{1}{\mu}.\nabla_{\dot{l}}\dot{l} + \frac{1}{\sqrt{\mu}}.\left(-\frac{1}{2\mu\sqrt{\mu}}.\mu'\right).u^{k'}.l_k = \frac{1}{\mu}.\nabla_{\dot{l}}\dot{l} -$
$\frac{1}{2.\mu^2}.\mu'.u^{k'}.l_k = \frac{1}{\mu}.\nabla_{\dot{l}}\dot{l} + \frac{1}{2}.\left(\frac{1}{\mu}\right)'.u^{k'}.l_k = \left(\frac{1}{\mu}.u^{k''} + \frac{1}{\mu}.\Gamma^k_{ij}.u^{i'}.u^{j'} + \frac{1}{2}.\left(\frac{1}{\mu}\right)'.u^{k'}\right).l_k .$

Hence,

$$|\nabla_t t|^2 = \mu.\left(\frac{1}{\mu}.u^{k''} + \frac{1}{\mu}.\Gamma^k_{ij}.u^{i'}.u^{j'} + \frac{1}{2}.\left(\frac{1}{\mu}\right)'.u^{k'}\right)^2 =$$
$$= \frac{1}{\mu}.|u''|^2 + \frac{1}{\mu}.v^2 + \frac{\mu}{4}.\left(\frac{1}{\mu}\right)' + \frac{2}{\mu}.< u'',v > + \left(\frac{1}{\mu}\right)'.< v, u' >, \tag{3}$$

where $v(s)$ is the vector function, defined in Lemma 3.1. Using the expression in the mentioned lemma and replacing the equalities

$$v^2 = \frac{4.u^2}{(1+u^2)^2}, \quad < u'', v > = \frac{2.< u,u'' >}{1+u^2} = \frac{u^{2''} - 2}{1+u^2},$$
$$< v, u' > = -\frac{2.< u,u' >}{1+u^2} = -\frac{u^{2'} - 2}{1+u^2}$$

in (3), after some reorganization and simplification we get $\tilde{\kappa}^2(s) = |\nabla_s t|^2$ and complete the proof. □

Using the properties of the Schwarzian derivative and the statement in theorem 3.1, we obtain that $\tilde{\kappa}^2(\sigma) = \frac{1}{\mu}.\kappa^2(\sigma) - 1 + 2.(S_s)(\sigma)$ for the arc-length parameter σ of γ. Let we denote by $\mathcal{M}(\sigma)$ the right side of the last expression. It is clear that the function

$$\mathcal{M}(\sigma) = \frac{1}{\mu}.\kappa^2(\sigma) - 1 + 2.(S_s)(\sigma), \tag{4}$$

defined for any Frenet plane curve c, is an invariant under the subgroup F_0 of the Möbius group in the plane.

Theorem 3.2 (Uniqueness theorem). *Let $I \subset R$ be an open interval and let $c_i: I \to R^2$, $i = 1, 2$ be two plane Frenet curves of the class C^2, parameterized by the same arc-length parameter σ of their stereographic images on the sphere S^2. Assume that the curves c_1 and c_2 have the same invariants $\mathcal{M}_i = \mathcal{M}_i(\sigma), i = 1, 2$, defined by (4), i.e. $\mathcal{M}_1(\sigma) = \mathcal{M}_2(\sigma)$ for any $\sigma \in I$. Then, there exists a transformation $f \in F_0$ such that $c_2 = f \circ c_1$.*

Proof. Since $\mathcal{M}_1(\sigma) = \mathcal{M}_2(\sigma)$ then the spherical pre-images γ_1 and γ_2 of the curves c_1 and c_2, respectively, via the stereographic projection $\pi: S^2 \to R^2$, have the same spherical curvatures. Therefore, there exists a rotation $\rho \in SO(3)$ such that $\gamma_2 = \rho(\gamma_1)$. Hence $c_2 = \pi(\gamma_2) = \pi \circ \rho(\gamma_1) = \pi \circ \rho \circ \pi^{-1}(c_1)$, where $f = \pi \circ \rho \circ \pi^{-1} \in F_0$ and the proof is completed. □

Theorem 3.2 (Existence theorem). *Let $g: I \to R$ be a function of class C^∞. Let e_1^0, e_2^0 be right-handed orthonormal pair of vectors at a point c_0 in the plane R^2. There exists a unique Frenet curve $c: I \to R^2$, which satisfies the conditions:*

(a) there is $\sigma_0 \in I$ such that $c(\sigma_0) = c_0$ and the Frenet frame of c at c_0 is e_1^0, e_2^0;

(b) for any $\sigma \in I$, $\mathcal{M}(\sigma) = g^2(\sigma)$, where $\mathcal{M}(\sigma)$ is the function, determined for the curve c by the equality (4).

Proof. Let l_0 be the corresponding point of the point c_0 via the stereographic projection $\pi: S^2 \setminus P \to R^2$, t_0 be the pre-image of the vector e_1^0 under the differential of the map π and n_0 be the unique vector that the triple l_0, t_0, n_0 is an orthonormal right-handed trihedron in R^3. Let us consider a matrix-valued function $\mathcal{X}(\sigma) = (l(\sigma), t(\sigma), n(\sigma))^T$. Solving the system of first order linear differential equations

$$\frac{d}{d\sigma}\mathcal{X}(\sigma) = A(\sigma)\mathcal{X}(\sigma)$$

with a given matrix $A(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & g \\ 0 & -g & 0 \end{pmatrix}$ and initial conditions l_0, t_0, n_0 we obtain a unique solution $\mathcal{X} = \mathcal{X}(\sigma)$, defined for all $\sigma \in I$, and $\mathcal{X}(\sigma_0) = (l_0, t_0, n_0)^T$ for some $\sigma_0 \in I$. It is routine to prove that the matrix $\mathcal{X}$ is orthogonal. This means that the vectors $l(\sigma), t(\sigma), n(\sigma)$ form an orthonormal triple in R^3 for any $\sigma \in I$. Let γ be a spherical curve, defined by the vector function $l = l(\sigma)$, and let $c = \pi(\gamma)$ be its corresponding plane curve. It is clear that the condition (a), in the statement of the theorem, is fulfilled. Since the spherical curvature $\tilde{\kappa}$ of γ is $\tilde{\kappa} = g$ then the proof of the condition (b) is obviously.

Now we can describe an algorithm of recovering a plane curve from one of its conformal invariants. First we fix a point c_0 and *a* right-handed orthonormal pair of vectors e_1^0, e_2^0 at a point c_0 in the plane R^2. According the proof of the theorem 3.2, for a given function $g: I \to R$ of class C^∞, we solve the system (5) with appropriate initial conditions. In the general case only a numerical solution of the system (5) is possible. This solution defines a spherical curve γ whose stereographic image $c = \pi(\gamma)$ is a plane curve, determined by the function g, up to transformation from the group F_n . We apply this algorithm in the computer system Mathematica.

Example. Let $\mathcal{M}(v) = \cos(v)$, $c_0 = (1, 0)$, $e_1^0 = (0, 1) \in R^2$. A plane curve with a conformal invariant $\mathcal{M}(\sigma) = \cos(\sigma)$ and its corresponding spherical curve under stereographic projection π are obtained by Mathematica and represented on Fig.2.

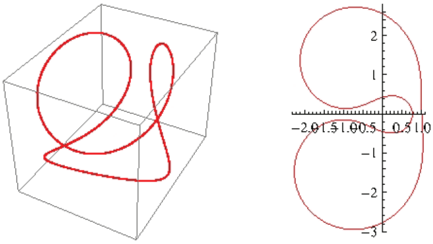

Figure 2

Other examples are considered in the demonstration [2], published on the site of Wolfram Demonstration Project. Moreover, it is shown here, how the curve depends on the parameters of the three-parameter subgroup F_0 of the Möbius group in the plane.

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